USE OF COMPUTER ALGEBRA SYSTEMS FOR SOLVING OF INDEFINITE INTEGRALS OF PROPER RATIONAL FUNCTIONS

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Abstract. *The calculations of indefinite integrals of proper rational functions are time consuming process. We have written a sequence of commands in Maple that do all calculations and present the intermediate results. This helps students to concentrated on the ideas, not on the algebraic calculations. It can also be used for checking of the calculations if the student is solving the problems by hand.*

Key words: Computer algebra system, Indefinite integral, Proper rational functions.

Mathematics Subject Classification: 97D40, 97D30

1. Introduction

The concept of mathematical competence now also includes skills to work with modern software systems. Using computer algebra systems (CAS) such as Maple, students develop programming and algorithmic thinking skills, expanding their digital competence beyond that of traditionally passive technologies. Using CAS also deepens students' mathematical knowledge and conceptual understanding of mathematics.

The development of information technology and computer systems provides a vast array of tools for solving both simple and difficult mathematical problems. Learners have access to a variety of software products. We will mention, for example, the mobile application PhotoMath, which makes it possible with just a picture of a problem to provide its solution step by step in different languages. The concept of teaching has undergone significant changes in recent years, because it is necessary for the teacher to comply with the available software applications. The future belongs to people, nations and countries who accept that a variety of mathematical applications are available to learners and succeed in embedding them in learning so that they contribute to the acquisition of more knowledge rather than being used as a substitute for knowledge.

We will illustrate the usage of the CAS Maple in teaching of indefinite integrals of proper rational functions in the classes of calculus.

2. Solving of indefinite integrals of proper rational functions

The calculations of indefinite integrals of proper rational functions are time consuming process. Let us consider the problem from Mathematical analysis (Integration of functions of one variable [2, §1.4].

Example 2.1. *Solve the integral*

$$\int \frac{xdx}{(x-1)(x+2)^2}.$$

Solution: The well-known procedure for calculating an indefinite integral of a proper fraction is related to the representation of the integral function

$$\frac{x}{(x-1)(x+2)^2}$$

as a sum of elementary fractions

$$\frac{x}{(x-1)(x+2)^2} = \frac{a}{x-1} + \frac{b}{x+2} + \frac{c}{(x+2)^2}.$$

After common denominator, we get the identity

$$\frac{x}{(x-1)(x+2)^2} = \frac{a(x+2)^2 + b(x-1)(x+2) + c(x-1)}{(x-1)(x+2)^2}.$$

Therefore, the numerators are identically equal, and we obtain the identity

$$x = a(x+2)^2+b(x-1)(x+2)+c(x-1) = (a+b)x^2+(4a+b+c)x+(4a-2b-c).$$

We equate the coefficients in front of the same powers and get the system

$$\left|\begin{array}{rcl} a+b &=& 0 \\ 4a+b+c &=& 1 \\ 4a-2b-c &=& 0 \end{array}\right.$$

After solving the system we get that $a = \frac{1}{9},\ b = -\frac{1}{9},\ c = \frac{2}{3}$ and we can solve the integral

$$\int \frac{xdx}{(x-1)(x+2)^2} = \frac{1}{9}\int \frac{dx}{x-1} - \frac{1}{9}\int \frac{dx}{x+2} + \frac{2}{3}\int \frac{dx}{(x+2)^2}.$$

Maple makes it easy to perform the series of calculations required to represent a proper fraction as a sum of elementary fractions. To work with polynomials, we run the package [1, 2]:

with(PolynomialTools):

We define the functions in the numerator and in the denominator in Example 2.1

$$P := x \to x; Q := x \to (x-1)\cdot(x+2)^2;$$

Maple returns

$$x \to x$$
$$x \to (x-1)(x+2).$$

We present the function $\dfrac{P(x)}{Q(x)}$ as a sum of elementary fractions

$$S := x \to factor\left(\frac{a}{x-1} + \frac{b}{x+2} + \frac{c}{(x+2)^2}\right);$$

Maple returns

$$x \to factor\left(\frac{a}{x-1} + \frac{b}{x+2} + \frac{c}{(x+2)^2}\right).$$

After a multiplication with the denominator we get the function in the numerator and we collect about the powers of x

$$R := x \to S(x)\cdot Q(x); collect(R(x), x);$$

Maple returns

$$x \to S(x)Q(x)$$
$$(a+b)x^2 + (4a+b+c)x + 4a - 2b - c.$$

We get the coefficients of the function $P(x) - R(x)$ and we count the number of coordinates of the Coefficient vector

$$w := CoefficientVector(P(x) - R(x), x); k := numelems(w);$$

Maple returns

$$\begin{bmatrix} -4a+2b+c \\ -4a-b-c+1 \\ -a-b \end{bmatrix}$$

3.

We get the system of equations that ensures $P(x) = R(x)$

$$\textbf{for } i \textbf{ from } 1 \textbf{ to } k \textbf{ do}$$
$$eq_i := w[i] = 0$$
$$\textbf{end do};$$

Maple returns

$$-4a + 2b + c = 0$$
$$-4a - b - c + 1 = 0$$
$$-a - b = 0.$$

By solving of the above system of equations

$$solve(\{eq[1], eq[2], eq[3]\}, \{a, b, c\});$$

Maple returns

$$\left\{a = \frac{1}{9}, b = -\frac{1}{9}, c = \frac{2}{3}\right\}.$$

We get the constants a, b and c and we can solve the problem.

Example 2.2. *Solve the integral*

$$\int \frac{(2x^4 + 3x^2 - x)dx}{(x-1)(1+x^2)^2}.$$

Solution: We define the functions in the numerator and in the denominator in Example 2.1

$$P := x \to 2x^4 + 3x^2 - x;$$
$$Q := x \to (x-1) \cdot (1+x^2)^2;$$

We present the function $\dfrac{P(x)}{Q(x)}$ as a sum of elementary fractions

$$S := x \to factor\left(\frac{a}{x-1} + \frac{bx+c}{x^2+1} + \frac{dx+e}{(x^2+1)^2}\right).$$

After applying the next commands

$$R := x \to S(x) \cdot Q(x); collect(R(x), x);$$
$$w := CoefficientVector(P(x) - R(x), x); k := numelems(w);$$
$$\textbf{for } i \textbf{ from } 1 \textbf{ to } k \textbf{ do}$$
$$eq_i := w[i] = 0$$
$$\textbf{end do};$$
$$solve(\{eq[1], eq[2], eq[3], eq[4], eq[5]\}, \{a, b, c, d, e\});$$

we get the constants $a = b = c = d = 1, e = 0$.

The next example requires a significant number of calculations.

Example 2.3. *Solve the integral*

$$\int \frac{(5x^9 + 18x^8 + 37x^7 + 48x^6 + 42x^5 + 18x^4 - 2x^3 - 11x^2 - 8x - 3)dx}{(x-1)(x+1)^3(1+x^2)(x^2+x+1)^2}.$$

Solution: We define the functions in the numerator and in the denominator in Example 2.1

$$P := x \to 5x^9 + 18x^8 + 37x^7 + 48x^6 + 42x^5 + 18x^4 - 2x^3 - 11x^2 - 8x - 3;$$
$$Q := x \to (x-1)(x+1)^3(1+x^2)(x^2+x+1)^2;$$

We present the function $\dfrac{P(x)}{Q(x)}$ as a sum of elementary fractions

$$S := x \to factor\left(\frac{a_1}{x-1} + \frac{a_2}{x+1} + \frac{a_3}{(x+1)^2} + \frac{a_4}{(x+1)^3} + \frac{a_5x+a_6}{x^2+1} + \frac{a_7x+a_8}{x^2+x+1} + \frac{a_9x+a_{10}}{(x^2+x+1)^2}\right).$$

After applying the next commands

$R := x \to S(x) \cdot Q(x); collect(R(x), x);$
$w := CoefficientVector(P(x) - R(x), x); k := numelems(w);$
for i **from** 1 **to** k **do**
$eq_i := w[i] = 0$
end do;
$solve(\{eq[i]\}_{i=1}^{10}, \{a_i\}_{i=1}^{10});$

we get the constants a_i for $i = 1, 2, \ldots, 10$.

The software Maple have a functions that present a rational function into elementary rational functions

$$convert\left(\frac{x}{(x-1)(x+2)^2}, parfrac, x\right);$$

$$convert\left(\frac{2x^4 + 3x^2 - x}{(x-1)(1+x^2)^2}, parfrac, x\right);$$

and

$$P := x \to 5x^9 + 18x^8 + 37x^7 + 48x^6 + 42x^5 + 18x^4 - 2x^3 - 11x^2 - 8x - 3;$$
$$convert\left(\frac{P(x)}{(x-1)(x+1)^3(1+x^2)(x^2+x+1)^2}, parfrac, x\right);$$

The software Maple can solve directly the integrals in the three examples with the command $\int f dx$

$$\int \frac{xdx}{(x-1)(x+2)^2}; \int \frac{(2x^4+3x^2-x)dx}{(x-1)(1+x^2)^2};$$
$$\int \frac{P(x)dx}{(x-1)(x+1)^3(1+x^2)(x^2+x+1)^2};$$

3. Conclusion

The presented sequence of commands can help students to check their intermediate results, when calculating by hand. By writing lines of commands and procedures, learners solidify their understanding of the techniques being studied.

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References

[1] A. Heck, *Introduction to Maple*, Springer, New York, 2003, ISBN: 978-1-4613-0023-6.

[2] B Zlatanov, *Real analysis: integral calculus of a function of one variable using algebraic computer systems*, University press "Paisius Hilendarski" Plovdiv, 2018, ISBN: 978-619-202-365-2 (in Bulgarian).

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