INVESTIGATIONS ON A DIFFERENTIAL SYSTEM WITH CORRECTION OF ZERNIKE–TYPE RADIAL POLYNOMIALS. SIMULATIONS

Evgenia Angelova, Valia Arnaudova, Todorka Terzieva, Anna Malinova

Abstract. *In this article we consider a new extended Lienard differential system with "corrections" of the Zernike–type radial polynomials* R_n^1*. The number and type of limit cycles in the light of Melnikov's consideration are also studied. Numerical examples, illustrating our results using CAS MATHEMATICA are given.*

Key words: Lienard system, Melnikov's approach, Zernike–type radial polynomials R_n^1.

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1. Introduction

Consider the Lienard system [2]

$$\begin{cases} \dfrac{dx}{dt} &= y - \epsilon\left(a_1 x + a_2 x^2 + \cdots + a_{2n+1} x^{2n+1}\right) \\ \dfrac{dy}{dt} &= -x \end{cases} \tag{1.1}$$

The *Melnikov polynomial* for the system (1.1) is defined as

$$P(r^2, n) = \frac{a_1}{2} + \frac{3}{8} a_3 \alpha^2 + \cdots + \binom{2n+2}{n+1} \frac{a_{2n+1}}{2^{2n+2}} r^{2n}. \tag{1.2}$$

It is known [3, 4] that the system (1.1) for sufficiently small $\epsilon \neq 0$ has at most n limit cycles asymptotic to circles of radii r_j, $j = 1, 2, \ldots, n$ as $\epsilon \to 0$ if and only if the nth degree polynomial $P(r^2, n)$ has n positive roots $r^2 = r_j^2$, $j = 1, 2, \ldots, n$.

Denote by R_n^1 the Zernike–type radial polynomials. In this paper we consider a extended Lienard–type system with the polynomial R_n^1. The number and type of limit cycles is also studied. Numerical examples, illustrating our results using *CAS MATHEMATICA* are given.

2. Main Results. Simulations

2.1. Extended Lienard–type planar system

The Zernike polynomials form a complete basis set of functions that are orthogonal over a circle of unit radius. The even Zernike polynomials are defined as (see [7, 8]) $Z_n^m(x,\phi) = R_n^m(x)\cos(m\phi)$, where R_n^m are the radial polynomials. In this Section we consider formally the following model:

$$\begin{cases} \dfrac{dx}{dt} &= y - \epsilon R_n^1(x) \\ \dfrac{dy}{dt} &= -x \end{cases} \tag{1.3}$$

where $\epsilon > 0$ and $n = 5,\ 7,\ 9,\ 11,\ \ldots$.

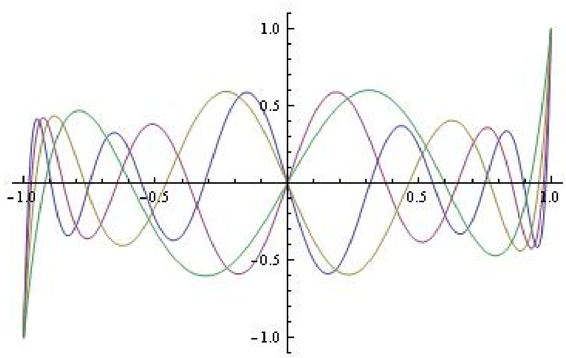

Figure 1. The polynomials $R_n^1(x)$ for $n = 5,\ 7,\ 9,\ 11$

For example we have (see Fig. 1).

$R_5^1(x) = 3x - 12x^3 + 10x^5$

$R_7^1(x) = -4x + 30x^3 - 60x^5 + 35x^7$

$R_9^1(x) = 5x - 60x^3 + 210x^5 - 280x^7 + 126x^9$

$R_{11}^1(x) = -6x + 105x^3 - 560x^5 + 1260x^7 - 1260x^9 + 462x^{11}$

Polynomials of this type can be used as correction factors in the Lienard differential system. The solutions of the system

$$\begin{cases} \dfrac{dx}{dt} &= y - \epsilon(R_5^1(x)) \\ \dfrac{dy}{dt} &= -x \end{cases} \tag{1.4}$$

for $\epsilon = 0.001$; $x_0 = 0.7$, $y_0 = 0.1$ are depicted on Fig. 2.

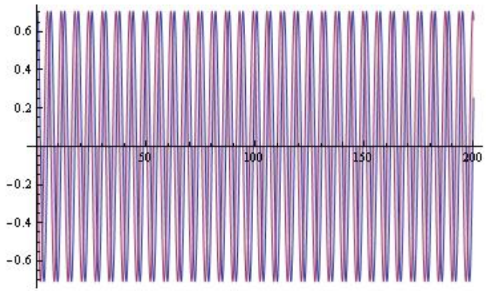

Figure 2. The solutions of the differential system (1.4)

2.2. The new model in the light of Melnikov's considerations

The case $n = 9$.

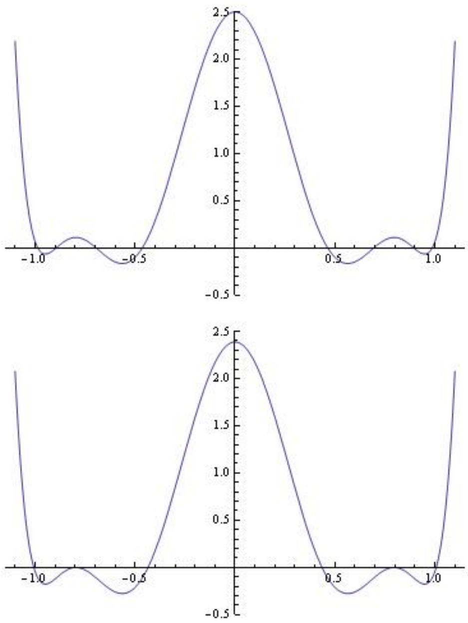

Figure 3. a) The Melnikov polynomial $P(r^2, 4)$ *for* $n = 9$ *and* $\mu = 5$ *(four limit cycles); b) The Melnikov polynomial* $P(r^2, 4)$ *for* $n = 9$ *and* $\mu = 4.775885349$ *(two simple limit cycles:* 0.435266, 1.00617 *and limit cycle* 0.7960 *with multiplicity – two)*

Consider the model

$$\begin{cases} \dfrac{dx}{dt} & = & y - \epsilon(\mu x - 60x^3 + 210x^5 - 280x^7 + 126x^9) \\ \dfrac{dy}{dt} & = & -x \end{cases} \tag{1.5}$$

where $\mu > 0$, $\epsilon > 0$.

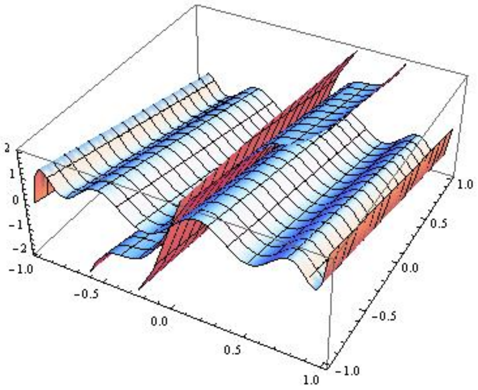

Figure 4. The catastrophe surface (x, y, p_2) for the following values of $p_2 = 5;\ 10;\ 15$

The following is valid

Proposition 2.1. *The Lienard–type system for $n = 9$, and for all sufficiently small $\epsilon \neq 0$ for $\mu = 4.775885349$ has two simple limit cycles: 0.435266, 1.00617 and limit cycle 0.7960 with multiplicity – two.*

Proof. For the Melnikov polynomial in r^2 (see Fig. 3) we have:

$$P(r^2, 4) = \frac{\mu}{2} - \frac{45}{2}r^2 + \frac{525}{8}r^4 - \frac{1225}{16}r^6 + \frac{3969}{128}r^8. \tag{1.6}$$

Evidently, for example $\mu = 4.775885349$ we have two simple limit cycles and limit cycle with multiplicity – two.

The catastrophe surfaces for $n = 9$, $(x, y, p_2) = p_2 x - 60x^3 + 210x^5 - 280x^7 + 126x^9 - y$ for the model is shown on Fig. 4.

Consider a Lienard system of type

$$\begin{cases} \dfrac{dx}{dt} &= y \\ \dfrac{dy}{dt} &= g(x) + \epsilon f(x) y \end{cases} \tag{1.7}$$

where $0 \leq \epsilon \leq 1$.

The solution of the system (1.7) for $x_0 = 0.5$, $y_0 = 0.5$, $\epsilon = 0.0001$, $g(x) = R_9^1(x)$, $f(x) = x - x^3 + x^5 - \frac{1}{7}x^7$ is visualized on Fig. 5. The solution of the system (1.7) for $x_0 = 0.7$, $y_0 = 0.3$, $\epsilon = 0.0001$, $g(x) = R_9^1(x)$, $f(x) = x - x^3 + x^5 - \frac{1}{7}x^7$ is depicted on Fig. 6.

For other results see [9]–[13].

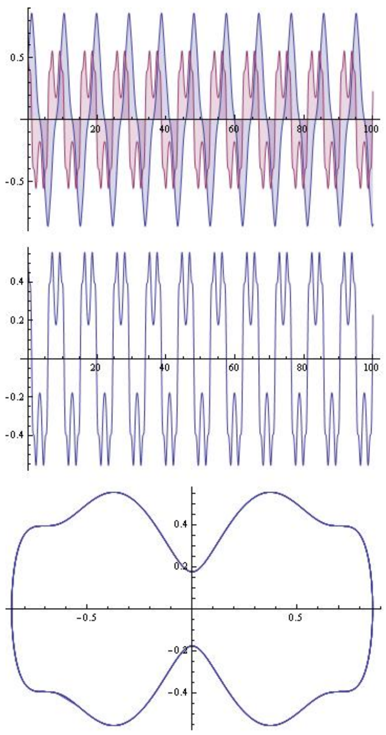

Figure 5. a) The solutions of the system (1.7) for $x_0 = 0.5$, $y_0 = 0.5$, $\epsilon = 0.001$, $\epsilon = 0.0001$, $g(x) = R_9^1(x)$, $f(x) = x - x^3 + x^5 - \frac{1}{7}x^7$; b) the y-component of the solution; c) the portrait

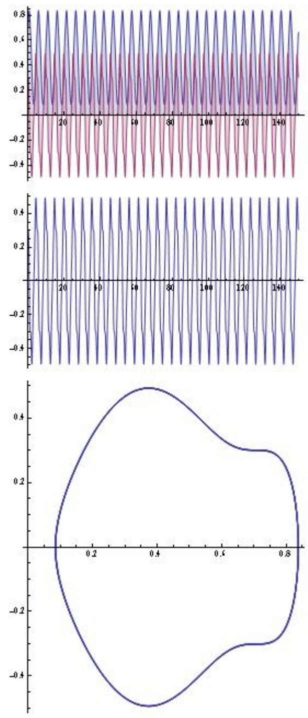

Figure 6. a) The solutions of the system (1.7) for $x_0 = 0.7$, $y_0 = 0.3$, $\epsilon = 0.001$, $\epsilon = 0.0001$, $g(x) = R_9^1(x)$, $f(x) = x - x^3 + x^5 - \frac{1}{7}x^7$; b) the y-component of the solution; c) the portrait

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Evgenia Angelova[1], Valia Arnaudova[2],
Todorka Terzieva[3], Anna Malinova[4,*]
[1,3,4] University of Plovdiv "Paisii Hilendarski",
Faculty of Mathematics and Informatics,
24 Tzar Asen Str., 4000 Plovdiv, Bulgaria
[2] Smolyan Filial, University of Plovdiv,
32 Dicho Petrov Str., 4700 Smolyan, Bulgaria
Corresponding author: malinova@uni-plovdiv.bg