COMMENTS ON A NEW EXPONENTIAL-X FAMILY AND APPLICATIONS TO ACTUARIAL SCIENCES

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Abstract. *Heavy-tailed distributions play a prominent role in actuarial and financial sciences. This talk deals with a new exponential-X (NE-X) family. The main purpose of this work is to continue studying the intrinsic properties of this heavy-tailed family, namely to study the "saturation" – d to the horizontal asymptote in the Hausdorff sense. The applicability of the proposed model is proved in simulation study to some insurance data and investigation of some actuarial metrics.*

Key words: new-exponential family, Weibull distribution, Hausdorff distance, Heaviside step function, Upper and lower bounds.

Mathematics Subject Classification: 41A46

1. Introduction

Most of the data sets in applied areas such as finance and actuarial sciences have heavy-tailed behavior with long right tail. Creating an approximation model with well-known classical distributions is quite a difficult task in some of the cases. For this reason researchers introduce a variety of new heavy-tailed distribution that are of interest to actuaries and finansists. One of the effective technique to develop a new one distributions is using general method transformed-transformer (T-X) proposed from Alzaatreh et al. [1]. Some important members of T-X family can be found in [2, 3]. In many recent papers new members of the T-X family are consider with their application to insurance for example see [4, 5, 6, 7, 8, 9].

In 2021 Ahmad, Mahmoudi, Roozegar, Alizadeh and Afify [10] proposed a new exponential-X (NE-X) family based on cumulative distribution function of the T-X family. The cumulative distribution function of NE-X family is given by

$$F_{NE-X}(t;\beta,\xi) = 1 - \left(\frac{1-G^{\beta}(t;\xi)}{\exp(G^{\beta}(t;\xi))}\right), \tag{1.1}$$

where $G(t;\xi)$ is the cumulative distribution function of the baseline distribution with parameter vector ξ and $\beta > 0$ is an additional parameter.

This paper deals with asymptotic behavior of some adaptive functions of the Hausdorff distance between Heaviside function and cumulative distribution function. One can see some similar investigation and approximation problems in related articles [11, 12, 13, 14, 15] and references therein.

Definition 1.1. *The shifted Heaviside step function is defined by*

$$h_{t_0}(t) = \begin{cases} 0 & \text{if } t < t_0, \\ [0,\, 1] & \text{if } t = t_0, \\ 1 & \text{if } t > t_0. \end{cases}$$

Definition 1.2. *[16, 17] The Hausdorff distance (the H–distance) $\rho(f,g)$ between two interval functions f, g on $\Omega \subseteq \mathbb{R}$, is the distance between their completed graphs $F(f)$ and $F(g)$ considered as closed subsets of $\Omega \times \mathbb{R}$. More precisely,*

$$\rho(f,g) = \max\{ \sup_{A\in F(f)} \inf_{B\in F(g)} ||A-B||, \sup_{B\in F(g)} \inf_{A\in F(f)} ||A-B|| \},$$

wherein $||.||$ is any norm in $\mathbb{R}^2$, e.g. the maximum norm $||(t,x)|| = \max\{|t|,|x|\}$; hence the distance between the points $A = (t_A, x_A)$, $B = (t_B, x_B)$ in $\mathbb{R}^2$ is $||A-B|| = \max(|t_A - t_B|, |x_A - x_B|)$.

The main purpose of this work is to continue studying the intrinsic properties of NE-X family, namely to study the "saturation" – d to the horizontal asymptote in the Hausdorff sense. We consider some applications in actuarial science.

2. Hausdorff approximation

Let us consider the "saturation" – d in the Hausdorff sense to the horizontal asymptote $a = 1$. For the function $F_{NE-X}(t;\beta,\xi)$ defined by (1.1) we have

$$F_{NE-X}(t_0;\beta,\xi) = \frac{1}{2} \quad \textit{with} \quad t_0 = G^{(-1)}\left(\left(1 - W\left(\frac{e}{2}\right)\right)^{1/\beta} \right),$$

where $W(x)$ is Lambert-W function.

Then the Hausdorff distance d between $F_{NE-X}(t;\beta,\xi)$ and the Heaviside function $h_{t_0}(t)$ at the "median level" satisfies the following nonlinear equation

$$F_{NE-X}(t_0+d;\beta,\xi)=1-d. \tag{1.2}$$

Next theorem gives upper and lower estimates for the Hausdorff approximation d.

Theorem 2.1. *Let*

$$A=1+\beta e^{z-1}\left(z+1\right)\left(1-z\right)^{1-\frac{1}{\beta}}G'\left(G^{(-1)}\left(\left(1-z\right)^{1/\beta}\right)\right)$$

with $z=W\left(e/2\right)$ and $2.1A>e^{1.05}$. Then for the Hausdorff distance d between shifted Heaviside function $h_{t_0}(t)$ and the cumulative distribution function $F_{NE-X}(t;\beta,\xi)$ defined by (1.1) *the following inequalities hold true:*

$$d_l=\frac{1}{2.1A}<d<\frac{\ln\left(2.1A\right)}{2.1A}=d_r.$$

Proof. We examine the following approximation of function

$$H(d)=F_{NE-X}(t;\beta,\xi)-1+d$$

as we use the function $T(d)=-\dfrac{1}{2}+Ad$. Indeed, from Taylor expansion we get $T(d)-H(d)=\mathcal{O}(d^2)$. The functions $T(d)$ and $H(d)$ are increasing. This means that $T(d)$ approximates $H(d)$ with $d\to 0$ as $\mathcal{O}(d^2)$. Note that if $2.1A>e^{1.05}$ holds then it is easy to show that $T(d_l)<0$ and $T(d_r)>0$. This completes the proof. □

2.1. NE-Weibull distribution

In this section we consider a spacial case of NE-X family as we use the classical Weibull distribution. The reader can formulate other special cases of proposed family using different baseline distributions with corresponding approximation problems. Let us recall that the Weibull distribution is associated with the following cdf and pdf functions:

$$F_{Wei}(t;\alpha,\gamma)=1-\exp\left(-\gamma t^{\alpha}\right)\quad\text{and}\quad f_{Wei}(t;\alpha,\gamma)=\alpha\gamma t^{\alpha-1}\exp\left(-\gamma t^{\alpha}\right).$$

Hence, from (1.1) we obtain new exponential-Weibull (NE-Weibull) distribution with cumulative distribution function given as

$$F_{NE-Weibull}(t;\beta,\alpha,\gamma) = 1 - \frac{1-(1-\exp(-\gamma t^{\alpha}))^{\beta}}{\exp\left((1-\exp(-\gamma t^{\alpha}))^{\beta}\right)}. \tag{1.3}$$

The Hausdorff distance d between $F_{NE-Weibull}(t;\beta,\alpha,\gamma)$ defined by (1.3) and the Heaviside function $h_{t_0}(t)$ satisfies the relation

$$F_{NE-Weibull}(t_0+d;\beta,\alpha,\gamma) = 1-d, \tag{1.4}$$

where $F_{NE-Weibull}(t_0;\beta,\alpha,\gamma) = \dfrac{1}{2}$ with

$$t_0 = \left(\frac{1}{\gamma}\log\left(\left(1-\left(1-W\left(\frac{e}{2}\right)\right)^{1/\beta}\right)^{-1}\right)\right)^{1/\alpha}.$$

Next corollary of Theorem 2.1 gives useful estimates for Hausdorff approximation d.

Corollary 2.1. *Let*

$$B = 1 - \frac{\alpha\beta\gamma\left(x^2-1\right)\left((1-x)^{-1/\beta}-1\right)\left(\frac{1}{\gamma}\log\left((1-(1-x)^{1/\beta})^{-1}\right)\right)^{\frac{\alpha-1}{\alpha}}}{2x}$$

with $x = W(e/2)$.

For the Hausdorff distance d between shifted Heaviside function $h_{t_0}(t)$ *and the cumulative distribution function* $F_{NE-Weibull}(t;\beta,\alpha,\gamma)$ *defined by* (1.3) *the following inequalities hold true for* $2.1B > e^{1.05}$*:*

$$d_l = \frac{1}{2.1B} < d < \frac{\ln(2.1B)}{2.1B} = d_r.$$

In Table 1 we present some computational examples for different values of parameters β, α and γ. We use Corollary 2.1 for computation the values of upper and lower estimates d_l and d_r. Several graphical representations are presented in Figure 1.

β	α	γ	d_l	d computed by (1.4)	d_r
3.92	12.63	2.12	0.046263	0.066574	0.142186
12.4	3.54	7.25	0.073519	0.104912	0.191901
0.74	0.58	1.81	0.079889	0.197346	0.201890
0.12	2.36	0.20	0.111268	0.241777	0.244324
1.64	0.95	0.83	0.313772	0.348894	0.363689

Table 1. Bounds for Hausdorff distance d computed by Corollary 2.1.

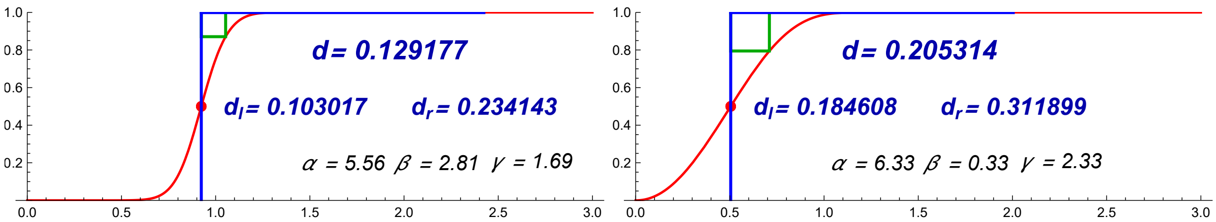

Figure 1. Approximation of NE-Weibull cumulative distribution function

3. Some applications to actuarial sciences

3.1. Approximating insurance data

The aplicability of the NE-Weibull distribution is presented in modeling insurance claims data. The data set is from the US Insurance Services Office (ISO) that comprises of 1500 non-life insurance claims of which are recorded the indemnity payment or loss, the allocated loss adjustment expense (ALAE) and the policy limit, i.e., the maximal claim amount [18].

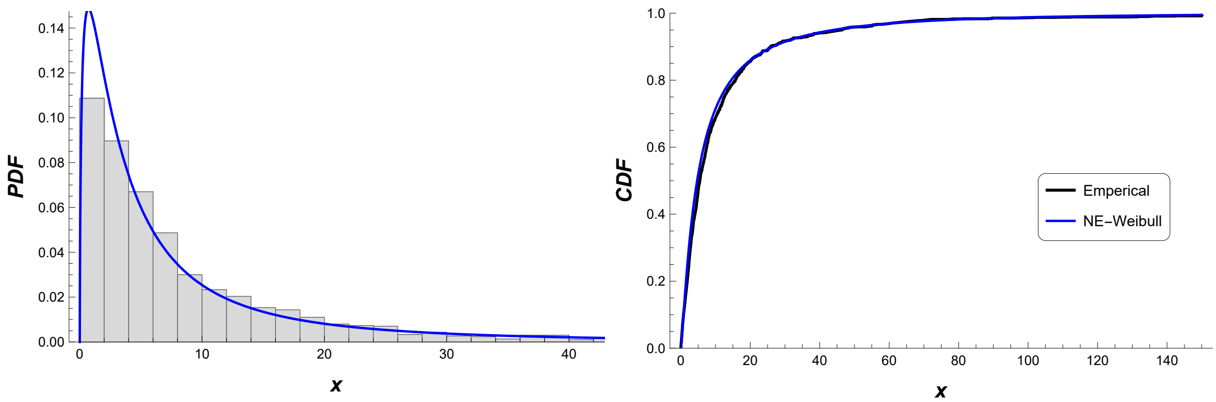

Figure 2. Approximation Loss and ALAE insurance data with NE-Weibull distribution

In Figure 2 we present an approximation model for consider data by NE-Weibull distribution with $\beta = 4.94$, $\alpha = 0.38$ and $\gamma = 0.87$. In this case the Hausdorff distance is $d = 0.471358$. Other example for modeling insurance data with NE-Weibull distribution can be found in [10].

3.2. Actural measures

Insurers and investors are interested in measuring insurance risk and risk in case of risky portfolios, respectively. In this section we consider two of the important measures *Value at Risk (VaR)* and *Tailed Value at Risk (TVaR)*. We calculate VaR and $TVaR$ for significance level q with

$$VaR(q) = \left(\frac{1}{\gamma}\log\left(1-(1-W(e-eq))^{-1/\beta}\right)\right)^{1/\alpha} \quad \text{and}$$

$$TVaR(q) = \frac{1}{1-q}\int_q^1 VaR(s)ds.$$

In Table 2 we camputate values of VaR and $TVaR$ for different significance levels q for NE-Weibull and classical Weibull distributions. We choose parameters such that cumulative distribution functions for these two distributions to have similar behaviour namely for NE-Weibull – $\alpha = 1.355$, $\gamma = 0.005$, $beta = 2.056$, and for $Weibull$ - $\alpha = 1.742$, $\gamma = 0.001$. Numerical rezults show that NE-Weibull have higher values and it can be more effective for modeling insurance data with heavy-tails. We also display these results graphically in Figure 3.

	NE-Weibull		**Weibull**	
level	$\alpha = 1.355, \gamma = 0.005, \beta = 2.056$		$\alpha = 1.742, \gamma = 0.001$	
q	VaR	TVaR	VaR	TVaR
0.85	78.0027	103.909	76.1727	95.9563
0.90	88.7211	114.358	85.1313	103.729
0.95	106.832	131.957	99.0139	116.051
0.999	201.598	223.257	159.949	172.55

Table 2. Values of the VaR and TVaR for NE-Weibull and Weibull distribution

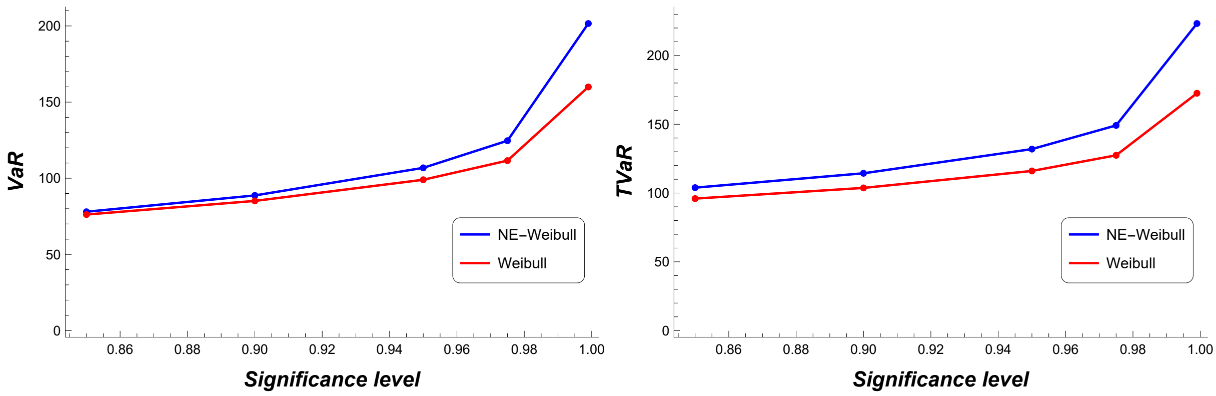

Figure 3. Graphical presentation for the values of the VaR and TVaR

Remark. Clarification of mathematical mechanism of the insurance and reinsurance for the proposed family can be done in the similar way as in [19, 20, 21].

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