RECENT PROGRESS IN THE DISTRIBUTION OF HARDY–LITTLEWOOD NUMBERS

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Abstract. *Hardy–Littlewood numbers are those integers that can be represented as the sum of a prime and a square. They are named after G.H. Hardy and J.E. Littlewood, who conjectured that every sufficiently large non square integer can be written in that manner. Here we give a brief account of the history and main developments concerning the Hardy–Littlewood numbers. We also discuss some recent results on gaps between consecutive Hardy–Littlewood numbers.*

Key words: Waring–Goldbach problem, Hardy–Littlewood circle method, sums of primes and squares.

Mathematics Subject Classification: 11P32, 11P55

1. Sums of primes and squares

In 1923, Hardy and Littlewood [6, Conjecture H] conjectured that if n is sufficiently large and not a square, then n can be represented as a sum of a prime and a square. We will follow the example of many other authors in calling such integers *Hardy–Littlewood numbers*. Hardy and Littlewood also proposed that the number $R(n)$ of such representations would be given asymptotically by

$$R(n) \sim \mathfrak{S}(n)\frac{\sqrt{n}}{\log n},$$

where

$$\mathfrak{S}(n) = \prod_{p>2}\left(1 - \frac{\left(\frac{n}{p}\right)}{p-1}\right),$$

with $\left(\frac{n}{p}\right)$ being the Legendre symbol. Although the Hardy–Littlewood conjecture is out of reach at the moment, important progress has been made on its resolution. Davenport and Heilbronn [4, 1937] proved that the conjecture is true for *almost all* natural numbers. More precisely, they showed that there exists some constant $A > 0$ such that all but

$O(x(\log x)^{-A})$ positive integers $n \leq x$ can be written as the sum of a prime and a square. This was shown to be true for arbitrary $A > 0$ by Miech [15, 1968]. For a given real number $x \geq 4$, let

$$E(x) = |\{n \leq x : n \text{ is not the sum of a prime and a square}\}|.$$

Then Miech's result implies that

$$E(x) \ll x(\log x)^{-A} \tag{1.1}$$

for any $A > 0$.

In 1975, Montgomery and Vaughan [19] proved that the cardinality of the exceptional set for the binary Goldbach conjecture, i.e. the set of even numbers not larger than a real number x which are not representable as the sum of two primes, is $O(x^{1-\delta})$ for some $\delta > 0$. Afterwards, a similar estimate for the exceptional set of the Hardy–Littlewood numbers was considered by several authors. Brünner, Perelli and Pintz [2, 1989] and A.I. Vinogradov [25, 1985] obtained, independently and using different techniques, that there exists an effectively computable constant $\delta > 0$ such that

$$E(x) \ll x^{1-\delta}.$$

Wang Tianze [26, 1995] showed that one can take $\delta = 0.01$. Li Hongze [12, 2003] set the current record by reaching $\delta = 0.018$.

The last results must be evaluated in light of the conditional estimates for $E(x)$ that have been proved assuming the validity of the Generalized Riemann Hypothesis (GRH). Mikawa [17, 1993] obtained, under GRH,

$$E(x) \ll x^{1/2}(\log x)^5. \tag{1.2}$$

Perelli and Zaccagnini [21, 1995] noted that by refining Mikawa's arguments one can replace the power 5 by $3+\varepsilon$. Suzuki [24, 2017] improved upon the last results by showing that

$$E(x) \ll x^{1/2}(\log x)^{3/2}(\log\log x)^4.$$

Another question concerning Hardy–Littlewood numbers is to ask whether they always exist in short intervals, or equivalently whether there exist long sequences of consecutive positive integers which cannot be written as the sum of a prime and a square. Mikawa [17, 1993] and Perelli and Pintz [20, 1995] showed independently that for any given A, $\varepsilon > 0$ and $7/24 + \varepsilon \le \theta \le 1$, one has

$$E(x + x^\theta) - E(x) \ll x^\theta (\log x)^{-A}.$$

The last bound may be regarded as a refinement of Miech's result (1.1), and should be compared with the conditional estimate (1.2). Afterwards, Languasco [10, 2004] was able to sharpen substantially the latter estimate by showing that, given $\varepsilon > 0$ and $7/24 + \varepsilon \le \theta \le 1$, one has

$$E(x + x^\theta) - E(x) \ll x^{\theta - \delta},$$

where $\delta = \delta(\theta) > 0$.

2. Sums of primes and squares of primes

In 1938, Hua [7] refined Hardy and Littlewood's conjecture by proposing that every sufficiently large even integer, satisfying some natural congruence conditions, could be represented as the sum of a prime and the square of another prime. Even though this problem is still unsolved, there has been some progress in the evaluation of its exceptional set. Let

$$\mathcal{H} = \{n \in \mathbb{N} \mid n \not\equiv 1 \,(\mathrm{mod}\, 3),\ 2|n\}.$$

Denote by $E^*(x)$ the number of integers $n \in \mathcal{H}$, with $n \le x$, which cannot be written as the sum of a prime and the square of a prime. Hua proved that

$$E^*(x) \ll x(\log x)^{-A} \tag{1.3}$$

with some $A > 0$. Schwarz [22, 1961] showed that (1.3) holds for any fixed $A > 0$. Bauer [1, 1999] and Leung and Liu [11, 1993] used the method of Montgomery and Vaughan [19, 1975] to prove independently that

$$E^*(x) \ll x^{1-\delta} \tag{1.4}$$

for some (very small) absolute constant $\delta > 0$.

Liu and Zhan [14, 1997] considered the short interval version of (1.3). They obtained that if A, $\varepsilon > 0$ and $7/16 + \varepsilon \le \theta \le 1$, then

$$E^*(x + x^\theta) - E^*(x) \ll x^\theta (\log x)^{-A}. \tag{1.5}$$

Afterwards, Kumchev and Liu [9, 2009] extended the admissible range for θ and proved that if $A > 0$ and $0.3275 \le \theta \le 1$, then the inequality (1.5) holds.

3. Gaps between consecutive terms of sequences

Denote by p_n the nth prime number. The problem of evaluating the difference $p_{n+1} - p_n$ has fascinated mathematicians for many years. It is a trivial consequence of the Prime Number Theorem that the expected average value of the difference $p_{n+1} - p_n$ is $\log p_n$. A well-known and still open conjecture of H. Cramér [3, 1936] asserts that

$$p_{n+1} - p_n \ll (\log p_n)^2.$$

Erdös [5, 1940] proposed the problem of estimating the second moment

$$\sum_{p_n \le x} (p_{n+1} - p_n)^2,$$

and Selberg [23, 1943] proved, assuming the Riemann hypothesis, that

$$\sum_{p_n \le x} (p_{n+1} - p_n)^2 \ll x(\log x)^3.$$

Now, let $\{g_n\}$ denote in ascending order the even integers that are representable as the sum of two primes. Then the famous binary Goldbach conjecture asserts that

$$g_{n+1} - g_n = 2 \tag{1.6}$$

for all n. Linnik [13, 1952] proved, on assuming the Riemann hypothesis, that

$$g_{n+1} - g_n \ll (\log g_n)^{3+\varepsilon}$$

for any $\varepsilon > 0$ and all n. Later, this result was improved by Kátai [8, 1967], and independently by Montgomery and Vaughan [19, 1975]; they showed, on the Riemann hypothesis, that the power $3 + \varepsilon$ can be replaced by 2.

The aforementioned works on gaps between consecutive primes inspired Mikawa [16, 1993] to consider the third moment of the difference $g_{n+1} - g_n$. He obtained that

$$\sum_{g_n \le x} (g_{n+1} - g_n)^3 \ll x(\log x)^{300}. \tag{1.7}$$

As a second result Mikawa proved that the asymptotic formula

$$\sum_{g_n \le x} (g_{n+1} - g_n)^\gamma = (2^{\gamma-1} + o(1))x \tag{1.8}$$

holds, provided that $0 \le \gamma < 3$.

These results show that (1.6) is true in a certain average sense. The proof of (1.7) utilizes a mean square estimate for the number of sums of two primes in short intervals. As for the proof of (1.8), it depends on two earlier results of Montgomery and Vaughan [19, 1975]—namely the estimate, mentioned previously, of the exceptional set for the Goldbach problem, and the estimate $\max_{g_n \le x}(g_{n+1} - g_n) \ll x^{7/72+\varepsilon}$.

4. Gaps between consecutive Hardy–Littlewood numbers

Let $\{h_n\}$ and $\{l_n\}$ denote in ascending order the integers that can be written as the sum of a prime and the square of an integer, and the sum of a prime and the square of a prime, respectively.

Recently, motivated by the works discussed earlier, Mikawa and the author [18] considered the second moment of the gaps between consecutive Hardy–Littlewood numbers, and established the following results.

Theorem 4.1 ([18]). *There exist some absolute constants $C_1 > 0$ and $C_2 > 0$ such that*

$$\sum_{h_n \le x} (h_{n+1} - h_n)^2 \ll x(\log x)^{C_1}$$

and

$$\sum_{l_n \le x} (l_{n+1} - l_n)^2 \ll x(\log x)^{C_2}. \tag{1.9}$$

The following asymptotic formula is obtained as a consequence of the latter bound.

Corollary 4.1 ([18]). *Suppose that $0 \le \gamma < 2$. Then one has the asymptotic formula*

$$\sum_{l_n \le x} (l_{n+1} - l_n)^\gamma = \tfrac{1}{6}(2^\gamma + 4^\gamma)\, x + o(x).$$

The proof of the corollary makes use of the estimate (1.4) of Bauer and Leung–Liu on the size of the exceptional set $E^*(x)$, the estimate (1.5) of Kumchev and Liu on the size of the exceptional set in short intervals, as well as the bound (1.9).

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