LIENARD SYSTEM WITH FIRST KIND CHEBYSHEV'S POLYNOMIAL–CORRECTION IN THE LIGHT OF MELNIKOV'S APPROACH. SIMULATIONS AND POSSIBLE APPLICATIONS

Vesselin Kyurkchiev, Anton Iliev, Asen Rahnev, Nikolay Kyurkchiev

Abstract. *In this article we consider a new extended Lienard–type system with "corrections" of the first kind Chebyshev's polynomial T_n. The number and type of limit cycles in the light of Melnikov's consideration are also studied. We will explicitly note that the $y(t)$-components of the differential systems can be used successfully in modeling and approximating of "U–shaped transfer functions" and some point sets in the field of signal theory. Numerical examples, illustrating our results using CAS MATHEMATICA are given.*

Key words: Lienard system, Melnikov's approach, first kind Chebyshev's polynomial T_n, extended model, number of limit cycles, "U–shaped transfer functions", level curves.

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1. Introduction

The Melnikov polynomial [1] for the system $\frac{dx}{dt} = y - \epsilon\left(a_1 x + a_2 x^2 + \cdots + a_{2n+1} x^{2n+1}\right)$; $\frac{dy}{dt} = -x$ is defined as

$$P(r^2, n) = \frac{a_1}{2} + \frac{3}{8} a_3 \alpha^2 + \cdots + \binom{2n+2}{n+1} \frac{a_{2n+1}}{2^{2n+2}} r^{2n}. \tag{1.1}$$

It is known [3, 4] that the system for sufficiently small $\epsilon \neq 0$ has at most n limit cycles asymptotic to circles of radii r_j, $j = 1, 2, \ldots, n$ as $\epsilon \to 0$ if and only if the nth degree polynomial $P(r^2, n)$ has n positive roots $r^2 = r_j^2$, $j = 1, 2, \ldots, n$. Denote by T_n the first kind Chebyshev's polynomial (see Fig. 1). The polynomials take part in some problems like antenna synthesis [5]–[7].

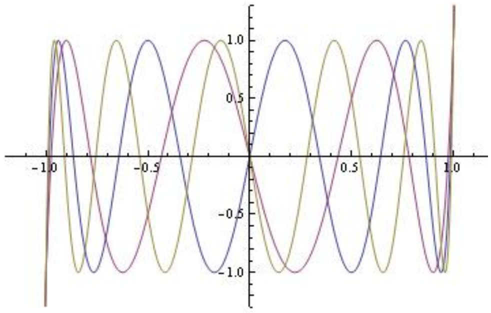

Figure 1. The polynomials $T_n(x)$ for $n = 7$, $n = 9$ and $n = 11$

2. Main Results. Simulations

In this Section we consider the following model of the type:

$$\begin{cases} \dfrac{dx}{dt} &= y - \epsilon T_n(x) \\ \dfrac{dy}{dt} &= -x \end{cases} \tag{1.2}$$

where $\epsilon > 0$ and $T_n(x)$ for $n = 5, 7, 9, \ldots$ is the Chebyshev's polynomial of the first kind. The simulation for user-selected coefficient $\epsilon = 0.001$ and $n = 9$, with the model (1.2) for $x_0 = 0$, $y_0 = 0.1$ is shown in Fig. 2.

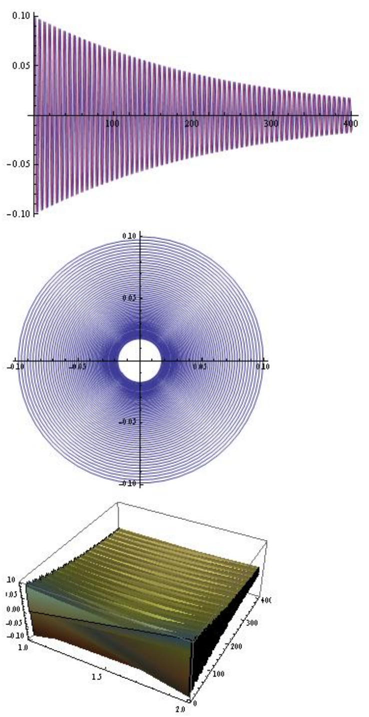

Figure 2. The solutions of the system for $\epsilon = 0.001$ and $n = 9$

The case $n = 9$

Consider the model for $\mu > 0$, $\epsilon > 0$

$$\begin{cases} \dfrac{dx}{dt} &= y - \epsilon(256x^9 - 576x^7 + 432x^5 - 120x^3 + \mu x) \\ \dfrac{dy}{dt} &= -x \end{cases} \tag{1.3}$$

The following is valid:

Theorem 2.1. *The Lienard–type system [2] (1.3) for* $n = 9$, *and for all sufficiently small* $\epsilon \neq 0$ *for* $\mu = 8.43986521$ *has two simple limit cycles* 0.388002, 1.0215 *and limit cycle* 0.808078 *with multiplicity – two.*

We note that for the polynomial $P(r^2, 4)$ (see Fig. 3) we have:

$$P(r^2, 4) = \frac{\mu}{2} - 45r^2 + 135r^4 - \frac{315}{2}r^6 + 63r^8. \tag{1.4}$$

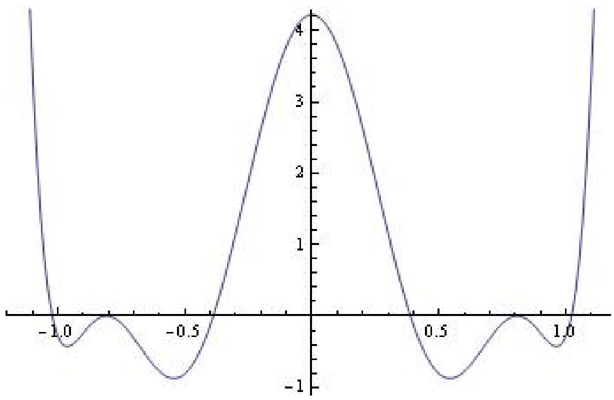

Figure 3. The Melnikov polynomial $P(r^2, 4)$ *for* $n = 9$ *and* $\mu = 8.43986521$

Related problems and possible applications

Consider a Lienard system for $0 \leq \epsilon \leq 1$

$$\begin{cases} \dfrac{dx}{dt} &= y \\ \dfrac{dy}{dt} &= g(x) + \epsilon f(x)y \end{cases} \tag{1.5}$$

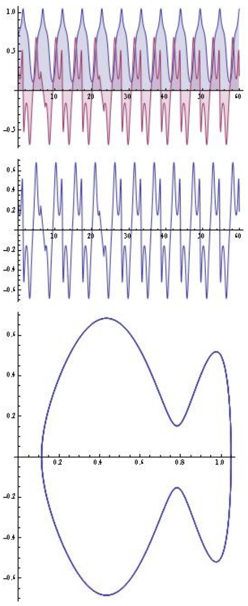

Figure 4. Example 1

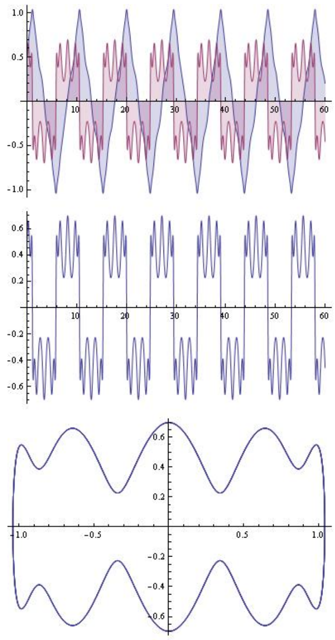

Figure 5. Example 2

Numerical examples

Example 1. The solution of the system (1.5) for $\epsilon = 0.0001$, $g(x) = T_7(x) = 64x^7 - 112x^5 + 56x^3 - 7x$, $f(x) = x - \frac{1}{3}x^3$ (see "correction" of the Van der Pol oscillator model) is visualized on Fig. 4.

Example 2. The solution of the system (1.5) for $\epsilon = 0.0001$, $g(x) = T_9(x) = 256x^9 - 576x^7 + 432x^5 - 120x^3 + 9x$, $f(x) = x - \frac{1}{3}x^3$ is visualized on Fig. 5.

We will explicitly note that the $y(t)$-components of the differential systems can be used successfully in modeling and approximating of "U–shaped transfer functions" (see Fig. 6) and some point sets in the field of signal theory. For other results see [8]–[12].

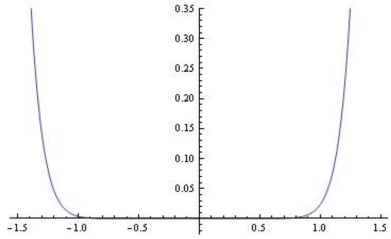

Figure 6. A typical "U–shaped transfer function"

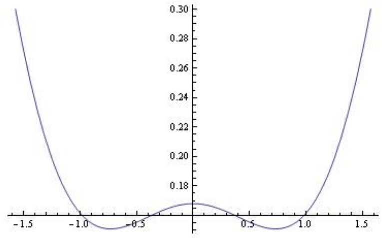

Figure 7. The model $y(\theta)$ (Example 1) for $x_0 = 0.5$, $y_0 = 0.5$ in $(-\frac{\pi}{2}, \frac{\pi}{2})$ for $a = 0.55$

Applications. We will consider two typical examples.

A). It is easy to take into account that the change of the variable t with $t = a\cos\theta$ in the y-component of the solution of the system (1.5) – (Example 1) leads to a good approximation of "U–shaped transfer function" at an appropriately selected interval. By varying the parameter a, a variety of interesting models are obtained (see Fig. 7).

B). The change of the variable t with $t = a\cos\theta$ in the y-component of the solution of the system (1.5) (Example 2) leads to a diagram – characteristic of a filter (see Fig. 8).

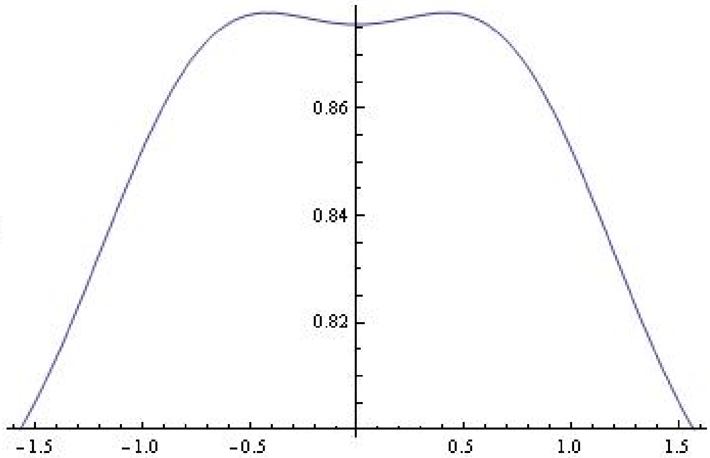

Figure 8. Application for modeling and analysis of filter–characteristics. The model $y(\theta)$ (Example 2) for $x_0 = 0.9$, $y_0 = 0.8$ in $(-\frac{\pi}{2}, \frac{\pi}{2})$ for $a = 0.11$

The level curves

In the last decades, the generalized polynomial Lienard differential systems have been studied intensively. For more details of existing important results on the topic: Limit cycles bifurcations of some generalized polynomial Lienard system see [13]–[17]. Consider the class of Lienard polynomial systems of the type

$$\begin{cases} \dfrac{dx}{dt} &= y \\ \dfrac{dy}{dt} &= T_7(x) + \epsilon(ax + bx^2 + cx^4 + dx^6)y \end{cases} \tag{1.6}$$

where $0 \le \epsilon < 1$; $T_7(x) = 64x^7 - 112x^5 + 56x^3 - 7x$ is the Chebyshev polynomial of the first kind and a, b, c, d are bounded parameters. Without going into details, we will note some interesting level curves. For $\epsilon = 0$ the system (1.6) is a Hamiltonian system with Hamiltonian

$$H(x,y) = H(x,y) = \frac{y^2}{2} - 8x^8 + \frac{56x^6}{3} - \frac{56x^4}{4} + \frac{7x^2}{2}.$$

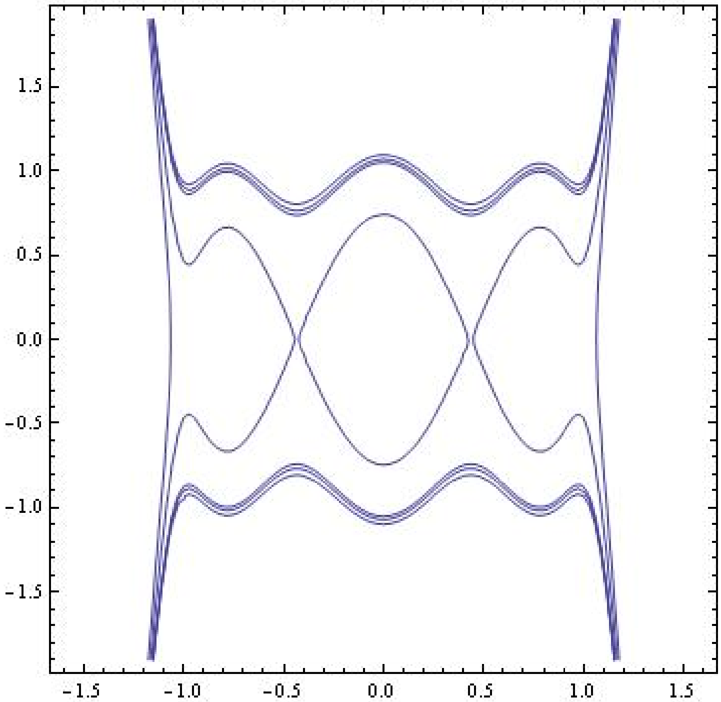

Figure 9. The level curves

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