MEASURE OF NONCOMPACTNESS FOR SOLVING ψ-CAPUTO-TYPE FRACTIONAL EVOLUTION EQUATIONS WITH NONDENSE DOMAIN

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Abstract. *In this manuscript, we establish a new existence theorem of solutions for evolution differential equations involving $\psi-$Caputo fractional derivative of order $0 < q < 1$ with nondense domaine. The existence result is proved by using Mönche's fixed point. As application, we conclude this paper by giving an illustrative example to demonstrate the applicability of the obtained result.*

Key words: $\psi-$fractional integral; $\psi-$Caputo fractional derivative; Carathéodory function, Mönche's fixed point.

Mathematics Subject Classification: 34K37, 26A33, 34A08

1. Introduction

The traditional integer calculus, is expanded by the fractional calculus, which has the characteristics of an infinite memory and is inherited. We recommend the reader consult the monographs for some basic findings in the theory of fractional calculus and fractional models [17, 20, 26, 28, 32]. In addition to the classical and fractional-order differential and integral operators, Almeida introduced in [9] the ψ-Caputo fractional derivative, which is another type of fractional derivative that defined by using a strictly increasing function, when a specific exponent function is included in the kernel operator.According to this idea, for specific selections of $\psi(t)$, a large class of well-known fractional derivatives, such as Caputo and Caputo-Hadamard, were found. Additionally, some intriguing information regarding the ψ-Caputo fractional derivative initial value and boundary value problems may be found in [4, 10, 13, 29]. The reader is urged to consult the references [7, 16, 21] for further information on fixed point theory, which is a highly helpful tool in the theory of the existence of solutions to functional and differential equations. which saw a lot of academics focus on the existence and uniqueness of solutions for differential equations involving various types of fractional derivatives under distinct boundary conditions. Additionally, a fascinating and effective technique for addressing the existence results for fractional differential equations is the measure of noncompactness. For instance, various writers have used the well-known Darbo fixed point theorem and the Monch fixed point theorem to obtain findings of existence for nonlinear integral equations [3, 8, 15]. The study of linked systems involving fractional differential equations, on the other hand, has become quite important. These systems appear in a variety of applied science applications. However, a study of coupled evolution systems with various derivatives, including Caputo derivative, are rare, if not nonexists yet. but there are a few in study of coupled evolution systems and albeit slowly, seen for example [1, 2, 22]. Consequently, the purpose of this work is to start to the growing field of knowledge in this area. To be realistic in this study, the authors in [33], explored the existence of mild solutions to starting value problems for fractional semilinear evolution equations with compact and noncompact semigroups on a local and global scale, as well as their uniqueness. They derive the fundamental solution's form from the Caputo fractional derivative-induced semigroup and ψ-function. Motivated by the above papers, we are devoted to establishing some results on the existence of solutions for a new coupled system of nonlinear fractional differential equations involving the ψ-Caputo derivative in abstract spaces. According to the authors' knowledge, no publication has examined nonlinear coupled evolution problem systems of ψ-Caputo differential equations with initial conditions in Banach Spaces. We shall then close this deficit. It is crucial to note that the solutions reported in this study are novel and produce a number of novel results as special instances for adequate parameter selection in the relevant problem. More specifically, we pose the following problem:

$$\begin{cases} D_{0^+}^{q;\psi} x(t) = Ax(t) + f(t,\ x(t),\ y(t)), \quad t \in \Delta = [0, T], \\ x(0) = x_0. \end{cases} \tag{1.1}$$

Where $T > 0$, $D_{0+}^{q,\psi}$ is the ψ-Caputo fractional derivative of order $q \in (0, 1]$, $f : [0,\ 1] \times X \times X \to X$, is a given functions satisfying some assumptions that will be specified later, X is a Banach space with norm $\|\cdot\|$ and $x_0 \in X$. and, A is a linear operator with nondense domaine.

The structure of this paper is as follows: in Sect. 2, we introduce definitions and preliminary results that we will need to prove our main results. In Sect. 3, we constructed the mild solution. In sec 4 we establish the existence of solutions for the problem (1.1). After that, we give a concrete example to illustrate our main results in Sect. 4 and the last section concludes this paper.

2. Preliminaries

In this section, we give some notations, definitions and results on ψ-fractional derivatives and ψ-fractional integrals, for more details we refer the reader to [9, 25].

Let $C(\Delta, X)$ be the Banach space of all continuous functions u from Δ into X with the supremum (uniform) norm:

$$\|u\|_\infty = \sup_{t\in\Delta}\{\|u(t)\}.$$

By $L^1(\Delta)$, we denote the space of Bochner-integrable functions $u : \Delta \to E$, with the norm:

$$\|u\|_1 = \int_0^T \|u(t)\|\mathrm{d}t.$$

Next, we define the Kuratowski measure of noncompactness and give some of its important properties.

Definition 2.1. *[14] The Kuratowski measure of noncompactness μ defined on bounded set S of Banach space X is:*

$$\mu(S) := \inf\{\epsilon > 0 : S = \bigcup_{k=1}^{n} S_k \textit{ and diam } (S_k) \leq \epsilon \textit{ for } k = 1,\, 2,\, \ldots,\, n\}.$$

The following properties about the Kuratowski measure of noncompactness are well known.

Proposition 2.1. *[14] Let X be a Banach space and $A,\ B \subset E$ be bounded. The following properties are satisfied:*

(1) $\mu(A) \leq \mu(B)$ if $A \subset B$;

(2) $\mu(A) = \mu(\mathrm{A}) = \mu(\overline{conv} A)$;

(3) $\mu(A) = 0$ if and only if A is relatively compact;

(4) $\mu(\lambda A) = |\lambda|\mu(A)_{\{}$ where $\lambda \in \mathbb{R}$;

(5) $\mu(A \cup B) = \max\{\mu(A),\ \mu(B)\}$*;*

(6) $\mu(A+B) \leq \mu(A) + \mu(B)$*, where* $A + B = \{w | w = a + b, a \in A, b \in B$*;*

(7) $\mu(A + x) = \mu(A)$ *for any* $x \in E$*.*

Lemma 2.1. *[19] Let* $V \subset C(\Delta,\ E)$ *be a bounded and equicontinuous subset. Then, the function* $t \to \mu(V(t))$ *is continuous on* Δ*:*

$$\mu_C(V) = \max_{t\in\Delta} \mu(V(t)) \quad \textit{and} \quad \mu\left(\int_\Delta u(s)\mathrm{d}s\right) \leq \int_\Delta \mu(V(s))\mathrm{d}s,$$

where $V(s) = \{u(s)\ :\ u \in V\}$*,* $s \in \Delta$*.*

Definition 2.2. *[34] A function* $f\ :\ [a,\ b] \times E \to E$ *is said to satisfy the Carathéodory conditions, if the following hold:*

– $f(t,\ u)$ *is measurable with respect to* t *for* $u \in X$*;*

– $f(t,\ u)$ *is continuous with respect to* $u \in X$ *for* $t \in \Delta$*.*

A useful fixed point result for our goals is the following

Theorem 2.1. *(Mönch's fixed point theorem [24]). Let* D *be a bounded, closed, and convex subset of a Banach space, such that* $0 \in D$*, and let* N *be a continuous mapping of* D *into itself. If the implication*

$$V = \overline{conv} N(V) \quad or \quad V = N(V) \cup \{0\} \Rightarrow \mu(V) = 0, \tag{1.2}$$

holds for every subset $V \subset D$*, then* $\mathcal{N}$ *has a fixed point.*

Now, we give some results and properties from the theory of fractional calculus. We begin by defining ψ-Riemann-Liouville fractional integrals, for more details we refer the reader to [9, 25]. In what follows:

In this section, we give some notations, definitions and results on ψ-fractional derivatives and ψ-fractional integrals, for more details we refer the reader to [9, 25].

- We denote by X a Banach space with the norm $\|\ .\ \|$.
- We denote by $\mathcal{C} := \mathrm{C}(\Delta, X)$ the Banach space of all continuous functions endowed with the topology of uniform convergence denoted by

$$\|\ x\ \|_\infty = \sup_{t\in\Delta} \|x(t)\|\ .$$

- We denote by B_r the closed ball centered at 0 with radius $r > 0$.

Definition 2.3. *[9] Let $q > 0$, $g \in L^1(\Delta, \mathbb{R})$ and $\psi \in C^n(\delta, \mathbb{R})$ such that $\psi'(t) > 0$ for all $t \in J$. The ψ-Riemann-Liouville fractional integral at order q of the function g is given by*

$$I_{0^+}^{q,\psi} g(t) = \frac{1}{\Gamma(q)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{q-1} g(s) ds. \tag{1.3}$$

Remark 2.1. *Note that if $\psi(t) = t$ and $\psi(t) = log(t)$, then the equation* (1.3) *is reduced to the Riemann-Liouville and Hadamard fractional integrals respectively.*

Definition 2.4. *[9] Let $q > 0$, $g \in C^{n-1}(J, \mathbb{R})$ and $\psi \in C^n(J, \mathbb{R})$ such that $\psi'(t) > 0$ for all $t \in J$. The $\psi-$Caputo fractional derivative at order q of the function g is given by*

$${}^C D_{0^+}^{q,\psi} g(t) = \frac{1}{\Gamma(n-q)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{n-q-1} g_\psi^{[n]}(s) ds, \tag{1.4}$$

where

$$g_\psi^{[n]}(s) = \left(\frac{1}{\psi'(s)} \frac{d}{ds} \right)^n g(s) \quad \text{and} \quad n = [q] + 1,$$

and $[q]$ denotes the integer part of the real number q.

Remark 2.2. *In particular, note that if $\psi(t) = t$ and $\psi(t) = log(t)$, then the equation* (1.4) *is reduced to the the Caputo fractional derivative and Caputo-Hadamard fractional derivative respectively.*

Remark 2.3. *In particular, if $q \in]0, 1[$, then we have*

$${}^C D_{0^+}^{q,\psi} g(t) = \frac{1}{\Gamma(q)} \int_0^t (\psi(t) - \psi(s))^{q-1} g'(s) ds$$

and

$${}^C D_{0^+}^{q,\psi} g(t) = I_{0^+}^{1-q,\psi} \left(\frac{g'(t)}{\psi'(t)} \right).$$

Proposition 2.2. *[9] Let $q > 0$, if $g \in C^{n-1}(J, \mathbb{R})$, then we have*

1) ${}^C D_{0^+}^{q,\psi} I_{0^+}^{q,\psi} g(t) = g(t)$.

2) $I_{0^+}^{q,\psi}\ {}^C D_{0^+}^{q,\psi} g(t) = g(t) - \sum_{k=0}^{n-1} \frac{g_{\psi}^{[k]}(0)}{k!}(\psi(t)-\psi(0))^k.$

3) $I_{a^+}^{q,\psi}$ *is linear and bounded from* $C(J,\mathbb{R})$ *to* $C(J,\mathbb{R})$.

Proposition 2.3. *[9] Let* $t > 0$ *and* $q,\ \beta > 0$*, then we have*

1) $I_{0^+}^{q,\psi}(\psi(t)-\psi(0))^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta+q)}(\psi(t)-\psi(0))^{q+\beta-1}.$

2) ${}^C D_{0^+}^{q,\psi}(\psi(t)-\psi(0))^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-q)}(\psi(t)-\psi(0))^{q-\beta-1}.$

3) ${}^C D_{0^+}^{q,\psi}(\psi(t)-\psi(0))^k = 0, \quad \forall k < n \in \mathbb{N}.$

Definition 2.5. *[33] Let* $x : \Delta \to X$ *be a function. The generalized Laplace transform of* x *is given by*

$$\mathcal{L}_{\psi}\{y(t)\}(s) := \widehat{x}(s) = \int_0^{\infty} \psi'(t) e^{-s(\psi(t)-\psi(0))} x(t)dt.$$

Definition 2.6. *[33] Let* f *and* g *be two functions which are piecewise continuous on* J *and of exponential order. The generalized* $\psi-$*convolution of* f *and* g *is defined by*

$$(f \underset{\psi}{*} g)(t) = \int_0^t f(s) g(\psi^{-1}(\psi(t)+\psi(0)-\psi(s)))\psi'(s)ds.$$

Lemma 2.2. *(See [33]). Let* $q > 0$ *and* y *be a piecewise continuous function on each interval* $[0,t]$ *and* $\psi(t)-$*exponential order. Then we have*

1. $\mathcal{L}_{\psi}\{I_{0^+}^{q,\psi} y(t)\}(s) = \frac{\widehat{y}(s)}{s^q}.$

2. $\mathcal{L}_{\psi}\{{}^C D_{0^+}^{q,\psi} y(t)\}(s) = s^q \left[\mathcal{L}_{\psi}\{y(t)\} - \sum_{k=0}^{n-1} s^{-k-1} f^{(k)}(0)\right]$, *where* $n = [q]+1$.

Definition 2.7. *(See [33]) Let* $\psi \in [0,\infty)$*. The one-sided stable probability density is defined by*

$$\omega_q(t) = \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}(\psi(t)-\psi(0))^{-qn-1}\frac{\Gamma(qn+1)}{n!}\sin(n\pi q).$$

It is easy to show that

Lemma 2.3. *The Laplace transform of $\omega_q(t)$ is given by*

$$\int_0^\infty e^{-\lambda(\psi(t)-\psi(0))}\omega_q(t)\psi'(t)dt = e^{-\lambda^q}.$$

Remark 2.4. *Note that for an abstract function $u\colon \Delta \to E$, the integrals which appear in the previous definitions are taken in Bochner's sense (see, for instance, [30]).*

3. Construction of mild solutions

In this section, we use the ψ-Laplace transform to construct the integral solution for the fractional evolution problem (1.1). For this purpose we need to prove to the following lemma.

Lemma 3.1. *The fractional evolution problem* (1.1) *is equivalent to the following integral equation*

$$x(t) = x_0 + I_{0^+}^{q,\psi}\big(Ax(t) + f\,(t, x(t))\,\big),\ \ \forall t \in \Delta. \tag{1.5}$$

Proof. Let x be a solution of the problem (1.1), then we apply the $\psi-$fractional integral $I_{0^+}^{q,\psi}$ on both sides of (1.1) we get

$$I_{0^+}^{q,\psi}\,{}^{C}D_{0^+}^{q,\psi}x(t) = I_{0^+}^{q,\psi}\big[Ax(t) + f\,(t, x(t))\,\big],$$

and by using Proposition 2.2 we obtain

$$x(t) - x(0) = I_{0^+}^{q,\psi}[Ax(t) + f(t, x(t))],$$

since $x(0) = x_0$, it follows that

$$x(t) = x_0 + \frac{1}{\Gamma(q)}\int_0^t \psi'(s)(\psi(t) - \psi(s))^{q-1}[Ax(s) + f\,(s, x(s))]ds.$$

Hence the integral equation (1.5) holds.

Conversely, by direct computation, it is clear that if x satisfies the integral equation (1.5), then the problem (1.1) holds which completes the proof. $\square$

Definition 3.1. *A function $x(t)$ is said to be an integral solution of* (1.1) *if*

1. $x : \Delta \to X$,
2. $I_{0^+}^{q,\psi} x(t) \in D(A), \ \forall t \in \Delta$,
3. $x(t) = x_0 + AI_{0^+}^{q,\psi} x(t) + I_{0^+}^{q,\psi} f\left((t, x(t))\right)$.

Remark 3.1. *We have the following remarks.*

1. *By using Proposition 2.2, we have* $I^{1,\psi} x(t) = I^{1-q,\psi} I_{0^+}^{q,\psi} x(t)$.
2. *If $x(t)$ is an integral solution of* (1.1)*, then* $I_{0^+}^{q,\psi} x(t) \in D(A), \ \forall t \in \Delta$, *which implies that*
$$I^{1,\psi} x(t) = I^{1-q,\psi} I_{0^+}^{q,\psi} x(t) \in D(A) \ \ \text{for} \ \ t \in \Delta.$$
3. *The limit* $\lim\limits_{h \to 0} \dfrac{1}{h} \displaystyle\int_t^{t+h} x(s)ds \in X_0$ *for* $t \in \Delta$ *shows that* $x(t) \in D(A)$.

Let A_0 be the part of A in $X_0 = \overline{D(A)}$ defined by

$$\begin{cases} D(A_0) = \{x \in D(A) : Ax \in \overline{D(A)}\}, \\ A_0 x = Ax. \end{cases}$$

We assume the following hypotheses throughout the rest of our paper.

(H_1) The linear operator $A : D(A) \subset X \to X$ satisfies the Hille-Yosida condition, that is, there exist two constant $\omega, M \in \mathbb{R}$ such that $(\omega, +\infty) \subseteq \psi(A)$ and

$$\left\| (\lambda I - A)^{-k} \right\| \leq \frac{M}{(\lambda - \omega)^k}, \quad \forall \lambda > \omega, k \geq 1.$$

(H_2) $T(t)$ is continuous in the uniform topology for $t > 0$.

Since the operator A_0 satisfies the Hille-Yosida condition, we can find the mild solution on $D(A_0)$. For this purpose, let us consider the following auxiliary problem:

$$\begin{cases} {}^C D_{0^+}^{q,\psi} x(t) = A_0 x(t) + g(t), \qquad t \in \Delta, \\ x(0) = x_0, \end{cases} \tag{1.6}$$

where g is a given continuous function.

Lemma 3.2. *Let $\lambda > \omega$, then the resolvent R_λ of A satisfies*

$$R_\lambda := (\lambda I - A)^{-1} = \int_0^\infty e^{-\lambda(\psi(t)-\psi(0))} T\left(\psi(t) - \psi(0)\right) \psi'(t) dt.$$

Proof. Let $x \in D(A)$. From (H_2) it follows that

$$\int_0^\infty e^{-\lambda(\psi(t)-\psi(0))} T\left(\psi(t) - \psi(0)\right) x\psi'(t) dt = \int_0^\infty e^{-\lambda t} T(t) dt = (\lambda I - A)^{-1} x.$$

That's true of all $x \in D(A)$, which implies the results. □

Proposition 3.1. *If the fractional integral equation*

$$x(t) = x_0 + \Phi(x) + I_{0^+}^{q,\psi}\big(A_0 x(t) + g(t)\big)$$

holds and g takes values in X_0, then we have

$$x(t) = S_{q,\psi}(t)(x_0) + \int_0^{\psi(t)-\psi(0)} K_{q,\psi}\left(\psi(t) - \psi(s)\right) g(s)\psi'(s) ds,$$

where

$$S_{q,\psi}(t) = I^{1-q,\psi} K_{q,\psi}(t),$$

and

$$K_{q,\psi}(t) = t^{q-1} \int_0^\infty q\left(\psi(\psi) - \psi(0)\right) \omega_q(\psi) T_{t^q(\psi(\psi)-\psi(0))} \psi'(\psi) d\psi.$$

Proof. Let $\lambda > 0$. From Lemmas 2.2 and 3.1 we have

$$\widehat{x} = \frac{1}{\lambda}(x_0) + \frac{1}{\lambda^q}\left(A_0 \widehat{x} + \widehat{g}\right),$$

we obtain

$$\widehat{x} = \lambda^{q-1}\left(\lambda^q I - A_0\right)^{-1}(x_0) + \left(\lambda^q I - A_0\right)^{-1} \widehat{g}.$$

Let's pose $I_1 = \lambda^{q-1}\left(\lambda^q I - A_0\right)^{-1}(x_0)$ and $I_2 = \left(\lambda^q I - A_0\right)^{-1} \widehat{g}$, i.e $I_2 = \displaystyle\int_0^{+\infty} e^{-\lambda^q s} T_s \widehat{g} ds.$

From Lemma 3.2, we get

$$\begin{aligned}
\lambda^{1-q} I_1 &= \left(\lambda^q I - A_0\right)^{-1}(x_0) \\
&= \int_0^{+\infty} e^{-\lambda^q(\psi(s)-\psi(0))} T_{\psi(s)-\psi(0)}(x_0)\, \psi'(s) ds \\
&= \int_0^{+\infty} q e^{-\left(\lambda(\psi(t)-\psi(0))\right)^q} T_{(\psi(t)-\psi(0))^q}(x_0)\, (\psi(t)-\psi(0))^{q-1} \psi'(t) dt \\
&= q \int_0^{\infty} \int_0^{+\infty} e^{-\lambda(\psi(t)-\psi(0))(\psi(s)-\psi(0))} \chi_q(s) \\
&\quad \times T_{(\psi(t)-\psi(0))^q}(x_0)\, (\psi(t)-\psi(0))^{q-1} \psi'(s)\psi'(t) dt ds \\
&= q \int_0^{\infty} \int_0^{+\infty} e^{-\lambda(\psi(t)-\psi(0))} \chi_q(s) \\
&\quad \times T_{\left(\frac{\psi(t)-\psi(0)}{\psi(s)-\psi(0)}\right)^q}(x_0) \frac{(\psi(t)-\psi(0))^{q-1}}{(\psi(s)-\psi(0))^q} \psi'(s)\psi'(t) dt ds \\
&= \int_0^{\infty} e^{-\lambda(\psi(t)-\psi(0))} \\
&\quad \times \Big[\int_0^{\infty} q\chi_q(s) \frac{(\psi(t)-\psi(0))^{q-1}}{(\psi(s)-\psi(0))^q} T_{\left(\frac{\psi(t)-\psi(0)}{\psi(s)-\psi(0)}\right)^q}(x_0)\, \psi'(s) ds \Big] \psi'(t) dt \\
&= \mathcal{L}_\psi\left(K_{q,\psi}\right)(\psi(t)-\psi(0))(x_0),
\end{aligned}$$

where

$$K_{q,\psi}(t) = \int_0^{\infty} q\chi_q(s) \frac{(\psi(t)-\psi(0))^{q-1}}{(\psi(s)-\psi(0))^q} T_{\left(\frac{\psi(t)-\psi(0)}{\psi(s)-\psi(0)}\right)^q}(x_0)\, \psi'(\psi) ds.$$

We can write

$$K_{q,\psi}(t) = t^{q-1} \int_0^{\infty} q\psi\omega_q(\psi) T_{t^q(\psi(\psi)-\psi(0))}\Big) \psi'(\psi) d\psi,$$

where

$$\omega_q(\psi) = (\psi(\psi)-\psi(0))^{\frac{-1}{q}} \chi_q\Bigg(\psi^{-1}\Bigg(\Bigg(\frac{1}{\psi(\psi)-\psi(0)}\Bigg)^{\frac{1}{q}} + \psi(0)\Bigg)\Bigg).$$

On other hand, we have

$$\mathcal{L}_\psi\left(\frac{(\psi(t)-\psi(0))^{-q}}{\Gamma(1-q)}\right)(\lambda) = \lambda^{q-1},$$

which implies that

$$I_1 = \mathcal{L}_\psi\left(\frac{(\psi(.)-\psi(0))^{-q}}{\Gamma(1-q)} * K_{q,\psi}(.)\right)(t) = \mathcal{L}_\psi\big(I^{1-q,\psi}K_{q,\psi}(t)\big).$$

Let us calculate I_2.

$$\begin{aligned}
I_2 &= \int_0^\infty e^{-\lambda^q(\psi(t)-\psi(0))}T_{\psi(t)-\psi(0)}\widehat{g}\psi'(t)dt \\
&= \int_0^\infty\int_0^\infty e^{-\lambda^q(\psi(t)-\psi(0))}e^{-\lambda(\psi(s)-\psi(0))}T_{\psi(t)-\psi(0)}g(s)\psi'(s)\psi'(t)dsdt \\
&= \int_0^\infty\int_0^\infty q\,(\psi(t)-\psi(0))^{q-1}\,e^{-\big(\lambda(\psi(t)-\psi(0))\big)^q}e^{-\lambda(\psi(s)-\psi(0))} \\
&\quad \times T_{(\psi(t)-\psi(0))^q}g(s)\psi'(s)\psi'(t)dtds \\
&= \int_0^\infty\int_0^\infty\int_0^\infty q\chi_q e^{-\big(\lambda(\psi(t)-\psi(0))(\psi(r)-\psi(0))\big)}e^{-\lambda(\psi(s)-\psi(0))}(r) \\
&\quad \times T_{(\psi(t)-\psi(0))^q}\,(\psi(t)-\psi(0))^{q-1}g(s)\psi'(r)\psi'(s)\psi'(t)dtdsdr \\
&= \int_0^\infty\int_0^\infty\int_0^\infty q\chi_q(r)e^{-\lambda(\psi(t)+\psi(s))} \\
&\quad \times T_{\left(\frac{\psi(t)-\psi(0)}{\psi(r)-\psi(0)}\right)^q}\frac{(\psi(t)-\psi(0))^{q-1}}{(\psi(r)-\psi(0))^q}g(s)\psi'(r)dr\psi'(s)ds\psi'(t)dt \\
&= \int_0^\infty e^{-\lambda(\psi(t)-\psi(0))} \\
&\quad \times\Big[\int_0^{\psi(t)-\psi(0)}\int_0^\infty q\chi_q(r)T_{\frac{(\psi(t)-\psi(0))^{q-1}}{(\psi(r)-\psi(0))^q}(\psi(r)-\psi(0))^q}\frac{(\psi(t)-\psi(s))^{q-1}}{(\psi(r)-\psi(0))^q} \\
&\quad \times g(s)\psi'(r)\psi'(s)drds\Big]\psi'(t)dt \\
&= \mathcal{L}_\psi\left(K_{q,\psi}\left(\psi(t)-\psi(s)\right)\right).
\end{aligned}$$

Thus $x(t)$ can be written as follows:

$$x(t) = S_{q,\psi}(t)(x_0) + \int_0^{\psi(t)-\psi(0)} K_{q,\psi}\left(\psi(t)-\psi(s)\right)g(s)\psi'(s)ds,$$

where $S_{q,\psi}(t) = I^{1-q,\psi}K_{q,\psi}(t)$, which completes the proof. $\square$

Remark 3.2. *From* (H_1) *we have* $\|\,R_\lambda\|\ \leq \dfrac{\lambda M}{\lambda-\omega}$, *then we get*

$$\lim_{\lambda\to+\infty}\|\,R_\lambda\|\ \leq M.$$

Proposition 3.2. *We assume that* (H_2) *holds, then*

1. *for a fixed* $t > 0, \{K_{q,\psi}(t)\}_{t>0}$ *and* $\{S_{q,\psi}(t)\}_{t>0}$ *are linear operators.*
2. *for* $x \in X_0$, *then* $\| K_{q,\psi}(t)x \| \leq \frac{t^{q-1}M}{\Gamma(1+q)}\|x\|$ *and* $\| S_{q,\psi}(t)x \| \leq \frac{M}{q} \| x \|$.
3. $\{K_{q,\psi}(t)\}_{t>0}$ *and* $\{S_{q,\psi}(t)\}_{t>0}$ *are strongly continuous.*

Proof. Since we have $\displaystyle\int_0^\infty q\psi\omega_q(\psi)\psi'(\psi)d\psi = \frac{1}{\Gamma(1+q)}$, then

$$\| K_{q,\psi}(t)x \| \leq \frac{t^{q-1}M}{\Gamma(1+q)}\| x \| .$$

From the above inequality it follows that

$$\begin{aligned}\Big\| S_{q,\psi}(\psi(t)-\psi(0))x \Big\| &= \Big\| I^{1-q,\psi}K_{q,\psi}(\psi(t)-\psi(0))x \Big\| \\ &\leq \frac{MI^{1-q,\psi}(\psi(t)-\psi(0))^{q-1}}{\Gamma(1+q)}\| x \| \\ &\leq \frac{M\Gamma(q)}{\Gamma(1+q)} \| x \| \\ &\leq \frac{M}{q}\| x \| ,\end{aligned}$$

which implies that $\| S_{q,\psi}(t)x \| \leq \frac{M}{q} \| x \|$.

Let $x \in X_0$ and $0 < t_1 < t_2 \leq T$, from a simple calculation, it follows that

$$\lim_{t_1\to t_2} \Big\| K_{q,\psi}(t_1)x - K_{q,\psi}(t_2)x\Big\| = 0 \text{ and } \lim_{t_1\to t_2} \Big\| S_{q,\psi}(t_1)x - S_{q,\psi}(t_2)x\Big\| = 0.$$

□

Lemma 3.3. *The integral equation of* (1.6) *is given by*

$$\begin{aligned}x(t) &= S_{q,\psi}(t)(x_0) \\ &\quad + \lim_{\lambda\to\infty}\int_0^{\psi(t)-\psi(0)} K_{q,\psi}\left(\psi(t)-\psi(s)\right)R_\lambda g(s)\psi'(s)ds.\end{aligned} \tag{1.7}$$

Proof. We have that

$$x_\lambda(t) = R_\lambda x(t)\ ,\ g_\lambda(t) = R_\lambda g(t)\ ,\ x_\lambda = R_\lambda x(0).$$

By applying R_λ to (1.6), we have

$$x_\lambda(t) = x_\lambda + A_0 I_{0^+}^{q,\psi} x_\lambda(t) + I_{0^+}^{q,\psi} g_\lambda(t)\ ,$$

hence

$$x_\lambda(t) = S_{q,\psi}(t)x_\lambda + \int_0^{\psi(t)-\psi(0)} K_{q,\psi}(\psi(t)-\psi(s))g_\lambda(s)ds,$$

since $x(t), x(0) \in X_0$, we have

$$x_\lambda(t) \to x(t),\ x_\lambda \to x(0),\ S_{q,\psi}(t)x_\lambda \to S_{q,\psi}(t)x(0),\ \text{as } \lambda \to +\infty.$$

Thus (1.7) holds. This completes the proof. □

Lemma 3.4. *Let $x \in X$ and $t \geq 0$, then*

$$\lim_{\lambda\to+\infty} \int_0^{\psi(t)-\psi(0)} K_{q,\psi}\left(\psi(s)-\psi(s)\right) R_\lambda x \psi'(s)ds$$

exists and the mapping

$$\eta_{q,\psi}(x) = \lim_{\lambda\to+\infty} \int_0^{\psi(t)-\psi(0)} K_{q,\psi}\left(\psi(s)-\psi(s)\right) R_\lambda x \psi'(s)ds$$

define a linear operator from X_0 into X_0.

Proof. Let $\Psi_{q,\psi}(t)$ be the following operator

$$\Psi_{q,\psi}(t)x_0 = \int_0^{\psi(t)-\psi(0)} K_{q,\psi}\left(\psi(s)-\psi(s)\right) R_\lambda x_0 \psi'(s)ds,$$

for $x_0 \in X_0$ and $t \geq 0$.

Then, the following operator

$$\varsigma_{q,\psi}(t) = (\lambda I - A)\Psi_{q,\psi}(t)(\lambda I - A)^{-1}, \quad \lambda > \omega,$$

extends $\Psi_{q,\psi}(t)$ from X_0 to X.

This definition is independent of λ due to resolvent identity. Since $\varsigma_{q,\psi}(t)$ maps X into X_0, then we have

$$\varsigma_{q,\psi}(t)x = \lim_{\lambda\to+\infty} R_\lambda \varsigma_{q,\psi}(t)x = \lim_{\lambda\to+\infty} \Psi_{q,\psi}(t)R_\lambda x.$$

This completes the proof. □

Lemma 3.5. *Let $x \in X_0$ and $t \geq 0$, then we have ${}^C D_{0^+}^{q,\psi}\Psi_{q,\psi}(t)x = S_{q,\psi}(t)x$ and $S_{q,\psi}(t)x = A\Psi_{q,\psi}(t)x + x$.*

Proof. The proof of this Lemma derived directly from the definitions of $S_{q,\psi}(t)$ and $\Psi_{q,\psi}(t)$ for $t \geq 0$. □

Lemma 3.6. *The following statements hold:*

(i) Let $x \in X$ and $t \geq 0$, then

$$I_{0^+}^{q,\psi}\varsigma_{q,\psi}(t)x \in D(A),$$

and

$$\varsigma_{q,\psi}(t)x = A(I_{0^+}^{q,\psi}\varsigma_{q,\psi}(\psi(t)-\psi(0))x) + \frac{(\psi(t)-\psi(0))^q}{\Gamma(1+q)}x.$$

(ii) If $x \in D(A)$, then

$$\varsigma_{q,\psi}(t)Ax + x = S_{q,\psi}(t)x.$$

Proof. To show (i), let $x \in X$ and $t \geq 0$, then we have

$$\begin{aligned}\zeta(t) &= \lambda I_{0^+}^{q,\psi}\Psi_{q,\psi}(t)(\lambda I - A)^{-1}x \\ &\quad + \frac{(\psi(t)-\psi(0))^q}{\Gamma(1+q)}(\lambda I - A)^{-1}x - \Psi_{q,\psi}(t)(\lambda I - A)^{-1}x.\end{aligned}$$

Clearly $\zeta(0) = 0$. From Lemma 3.5 we have

$$\begin{aligned}{}^C D_{0^+}^{q,\psi}\zeta(t) &= \lambda\Psi_q(t)(\lambda I - A)^{-1}x + (\lambda I - A)^{-1}x - c_{{}^C D_{0^+}^{q,\psi}\Psi_{q,\psi}(t)(\lambda I} - A)^{-1}x \\ &= \lambda\Psi_{q,\psi}(t)(\lambda I - A)^{-1}x + (\lambda I - A)^{-1}x - S_{q,\psi}(t)(\lambda I - A)^{-1}x \\ &= \lambda\Psi_{q,\psi}(t)(\lambda I - A)^{-1}x + (\lambda I - A)^{-1}x - A\Psi_{q,\psi}(t)(\lambda I - A)^{-1}x \\ &\quad - (\lambda I - A)^{-1}x\end{aligned}$$

$$
\begin{aligned}
&= \lambda\Psi^0_{q,\psi}(t)(\lambda I - A)^{-1}x - A\Psi_{q,\psi}(t)(\lambda I - A)^{-1}x \\
&= (\lambda I - A)\Psi_{q,\psi}(t)(\lambda I - A)^{-1}x \\
&= \varsigma_{q,\psi}(t)x.
\end{aligned}
$$

It follows that

$$\zeta(t) = I^{q,\psi}_{0^+}\varsigma_{q,\psi}(t)x + \zeta(0) = I^{q,\psi}_{0^+}\varsigma_{q,\psi}(t)x,$$

and

$$
\begin{aligned}
(\lambda I - A)\zeta(t) &= (\lambda I - A)I^{q,\psi}_{0^+}\varsigma_{q,\psi}(t)x \\
&= \lambda I^{q,\psi}_{0^+}\varsigma_{q,\psi}(t)x + \frac{(\psi(t)-\psi(0))^{q,\psi}}{\Gamma(1+q)}x - \varsigma_{q,\psi}(t)x.
\end{aligned}
$$

Thus

$$\varsigma_{q,\psi}(t)x = A(I^{q,\psi}_{0^+}\varsigma_{q,\psi}(t)x) + \frac{(\psi(t)-\psi(0))^{q}}{\Gamma(1+q)}x.$$

Now, we prove (ii). Let $x \in D(A)$, it follows from Lemmas 3.4 and 3.5 that

$$
\begin{aligned}
\varsigma_{q,\psi}(t)Ax &= \lim_{\lambda\to+\infty}\int_0^{\psi(t)-\psi(0)} K_{q,\psi}(\psi(s)-\psi(0))R_\lambda Ax\psi'(s)ds \\
&= \lim_{\lambda\to+\infty} A_0\int_0^{\psi(t)-\psi(0)} K_{q,\psi}(s)R_\lambda x\psi'(s)ds \\
&= A_0\Psi_{q,\psi}(t)x = S_{q,\psi}(t)x - x.
\end{aligned}
$$

This completes the proof. □

Theorem 3.1. *The mild solution of the evolution problem* (1.6) *is given by*

$$x(t) = S_{q,\psi}(x_0) + \lim_{\lambda\to\infty}\int_0^t K_{q,\psi}\left(\psi(t)-\psi(s)\right)R_\lambda g(s)ds.$$

Proof. The proof is given in several steps:

1) **Step 1:** Let g be always differentiable, then for $t \in \Delta$, we have.

$$\begin{aligned}
x_\lambda(t) &= \int_0^{\psi(t)-\psi(0)} K_{q,\psi}(s) R_\lambda g(s) \psi'(s) ds \\
&= \int_0^{\psi(t)-\psi(0)} K_{q,\psi}(s) R_\lambda (g(0) + \int_0^s g'(r) dr) \psi'(s) ds \\
&= \int_0^{\psi(t)-\psi(0)} K_{q,\psi}(s) R_\lambda g(0) \psi'(s) ds \\
&\quad + \int_0^{\psi(t)-\psi(0)} K_{q,\psi}(s) R_\lambda \int_0^s g'(r) dr) \psi'(s) ds \\
&= \Psi_{q,\psi}(t) R_\lambda g(0) + \int_0^t \varsigma^0_{q,\psi}(\psi(t)-\psi(r)) R_\lambda g'(r) dr.
\end{aligned}$$

By Lemma 3.5 for $t \in \Delta$, we obtain

$$\begin{aligned}
x(t) &= \lim_{\lambda \to +\infty} x_\lambda(t) \\
&= \varsigma_{q,\psi}(t) g(0) + \int_0^t \varsigma_{q,\psi}(\psi(t)-\psi(r)) g(r) dr \\
&= A(I^{q,\psi}_{0^+} \varsigma_{q,\psi}(t) g(0)) + \frac{(\psi(t)-\psi(0))^q}{\Gamma(1+q)} g(0) \\
&\quad + \int_0^t [A(I^{q,\psi}_{0^+} \varsigma_{q,\psi}(\psi(t)-\psi(r))) + \frac{(\psi(t)-\psi(r))^q}{\Gamma(1+q)}] g'(r) dr \\
&= A[I^{q,\psi}_{0^+} \Phi_{q,\psi}(t) f(0) + \int_0^t I^{q,\psi}_{0^+} \Phi_{q,\psi}(\psi(t)-\psi(r)) g'(r) dr] \\
&\quad + \frac{(\psi(t)-\psi(0))^q}{\Gamma(1+q)} g(0) + \frac{1}{\Gamma(1+q)} \int_0^t (\psi(t)-\psi(r))^q g'(r) dr \\
&= A[I^{q,\psi}_{0^+} \varsigma_{q,\psi}(t) g(0) + I^{q,\psi}_{0^+} (\int_0^t \Phi_{q,\psi}(\psi(t)-\psi(r)) g'(r) dr)] \\
&\quad + \frac{(\psi(t)-\psi(0))^q}{\Gamma(1+q)} g(0) + \frac{1}{\Gamma(1+q)} \int_0^t (\psi(t)-\psi(r))^q g'(r) dr \\
&= A(I^{q,\psi}_{0^+} x(t)) + I^{q,\psi}_{0^+} g(t) \ .
\end{aligned}$$

2) **Step 2:** Now, we approach g through continuously differentiable functions g_n such that:

$$\sup_{t \in \Delta} \|g(t) - g_n(t)\| \to 0, \ \text{as } n \to \infty.$$

Letting

$$x_n(t) = \lim_{\lambda\to\infty} \int_0^t K_{q,\psi}(\psi(s)) R_\lambda g_n(s) ds,$$

we have

$$x_n(t) = A(I_{0^+}^{q,\psi} x_n(t)) + I_{0^+}^{q,\psi} g_n(t)\,.$$

Hence

$$\begin{aligned} \| \, x_n(t) - x_m(t) \| &= \left\| \lim_{\lambda\to\infty} \int_0^t K_{q,\psi}(s) R_\lambda [g_n(s) - g_m(s)] ds \right\| \\ &\leq \frac{M(\psi(T) - \psi(0))^q}{\Gamma(q)} \| \, g_n - g_m \| \;, \end{aligned}$$

which implies that $\{x_n\}$ is a Cauchy sequence and its limit is denoted by $x(t)$, thus we obtain $x(t) = A(I_{0^+}^{q,\psi} x(t)) + I_{0^+}^{q,\psi} f(t)$ for $t \in \Delta$. This completes the proof.

□

Corollary 3.1. *By using definition 3.1, remark 3.1 and theorem 3.1 we can give the mild solution of the fractional evolution problem* (1.1) *as follows:*

$$x(t) = S_{q,\psi}\left(x_0\right) + \lim_{\lambda\to\infty} \int_0^t K_{q,\psi}\left(\psi(t) - \psi(s)\right) R_\lambda f(s, x(s)))\psi'(s) ds.$$

4. Main results

Now, we shall present our main result concerning the existence of solutions of problem (1.1). Let us introduce the following hypotheses:

(H_3) The function f: $\Delta \times X \to X$ satisfy Carathéodory conditions.

(H_4) There exist $\mu_f \in L^\infty(\Delta,\ \mathbb{R}_+)$, and a continuous nondecreasing function ϕ: $\mathbb{R}_+ \to \mathbb{R}_+$, such that $\|f(t,u)\| \leq \mu_f(t)\phi(\|u\|)$ for a.e. $t \in \Delta$ and each $u \in X$.

(H3) For each bounded set $D \subset X$, and each $t \in \Delta$, the following inequality holds:

$$\mu(f(t,\ D)) \leq \mu_f(t)\mu(D).$$

In the following, for computational convenience, we put:

$$L = \frac{(\psi(T) - \psi(0))^q}{\Gamma(1+q)}, \quad \mu_f^* = \sup_{t\in\Delta} \mu_f(t).$$

Now, we shall prove the following theorem concerning the existence of solutions of problem (1.1).

Theorem 4.1. *Assume that the hypotheses* $(H1)-(H3)$ *are satisfied. If*

$$L\mu_f^* < 1, \tag{1.8}$$

then the problem (1.1) *has at least one solution defined on* Δ

Proof. Define the operator $\mathcal{T} : C(\Delta, X) \to C(\Delta, X)$, by:

$$(\mathcal{T}(x))(t) = S_{q,\psi}(x_0) + \lim_{\lambda\to\infty} \int_0^t K_{q,\psi}(\psi(t)-\psi(s))\, R_\lambda f(s,x(s)))\psi'(s)ds, \ t \in \Delta. \tag{1.9}$$

It is obvious that $\mathcal{T}$ is well defined due to (H_1) and (H_2). Then, the fractional integral equation (7) can be written as the following operator equation:

$$x = \mathcal{T}(x). \tag{1.10}$$

Thus, the existence of a solution for equation (1.1) is equivalent to the existence of a fixed point for operator $\mathcal{T}$ which satisfies operator equation (1.10). Define a bounded closed convex set:

$$B_R = \{x \in C(\Delta, X) \ : \ \|x\| \le R\},$$

with $R > 0$, such that:

$$R \ge M\left(\frac{\|x_0\|}{q} + \phi(R)\right).$$

To satisfy the hypotheses of Mönch's fixed point theorem, we split the proof into four steps.

Step 1. The operator $\mathcal{T}$ maps the set B_R into itself. Let $x \in B_R$. Then, for each $t \in \Delta$, we have:

$$\|(\mathcal{T}(x))(t)\| \le \|S_{q,\psi}\|\|x_0\| + \int_0^t \frac{\psi'(s)(\psi(t)-\psi(s))^{q-1}}{\Gamma(q)}\|K_{q,\psi}\|\|f(s,x(s))\|\mathrm{d}s.$$

By using (H_2) and (H_4), for each $t \in \Delta$, we have:

$$\|f(t, x(t))\| \leq \mu_f(t)\phi(\|x(t)\|) \leq \mu_f^*\phi(\|x(t)\|.$$

Hence:

$$\|(\mathcal{T}(x))\|_\infty \leq \frac{M}{q}\|x_0\| + ML\mu_f^*\phi(R),$$

but, $L\mu_f^* < 1$, which implies that

$$\leq M\left(\frac{\|x_0\|}{q} + \phi(R)\right).$$

$$\leq R.$$

This proves that $\mathcal{T}$ transforms the ball B_R into itself.

Step 2. The operator $\mathcal{T}$ is continuous. Consider a sequence $\{x_n\} \in B_R$, such that $x_n \to x$ as $n \to \infty$. We need to show that $\|\mathcal{T}x_n - \mathcal{T}x\| \to 0$ as $n \to \infty$. On one hand, it is easy to see that $f(s, x_n(s)) \to f(s, x(s))$, as $n \to +\infty$, due to the Carathéodory continuity of f. On the other hand, taking (H_2) into consideration, we get the following inequality:

$$\begin{aligned}&\psi'(s)(\psi(t) - \psi(s))^{q-1}\|f(s, x_n(s)) - f(s, x(s))\|\\ &\leq 2\mu^* f\phi(R)\psi'(s)(\psi(t) - \psi(s))^{q-1}.\end{aligned}$$

We notice that since the function $s \mapsto 2\mu_f^* 1\phi(R)\psi'(s)(\psi(t) - \psi(s))^{q-1}$ is Lebesgue integrable over $[a, t]$. This fact together with the Lebesgue dominated convergence theorem implies that:

$$\int_a^t \frac{\psi'(s)(\psi(t) - \psi(s))^{q-1}}{\Gamma(q)}\|f(s, x_n(s)) - f(s, x(s))\|\mathrm{d}s \to 0, \text{ as } n \to +\infty.$$

It follows that:

$$\|(\mathcal{T}(x_n))(t) - (\mathcal{T}(x))(t)\| \to 0 \text{ as } n \to +\infty \text{ for any } t \in \Delta.$$

Therefore, we get that:

$$\|(\mathcal{T}(x_n)) - (\mathcal{T}(x))\| \to 0 \text{ as } n \to +\infty,$$

which implies the continuity of the operator $\mathcal{T}$.

Step 3. The operator $\mathcal{T}$ is equicontinuous. For any $a < t_1 < t_2 < b$ and $x \in B_R$, we get:

$$\begin{aligned}
&\|(\mathcal{T}(x))(t_2) - (\mathcal{T}(x))(t_1)\| \\
&\leq \int_0^{t_1} \frac{\psi'(s)[(\psi(t_1)-\psi(s))^{q-1} - (\psi(t_2)-\psi(s))^{q-1}]}{\Gamma(q)} \|f(s,x(s))\| \mathrm{d}s \\
&\quad + \int_{t_1}^{t_2} \frac{\psi'(s)(\psi(t_2)-\psi(s))^{q-1}}{\Gamma(q)} \|f(s,x(s))\| \mathrm{d}s \\
&\leq \frac{\mu_f^* \phi_i(R)}{\Gamma(q+1)} [(\psi(t_1)-\psi(a))^q + 2(\psi(t_2)-\psi(t_1))^q - (\psi(t_2)-\psi(a))] \\
&\leq \frac{2\mu_f^* \phi(R)}{\Gamma(q+1)} (\psi(t_2)-\psi(t_1))^q,
\end{aligned}$$

where we have used the fact that $(\psi(t_1)-\psi(a))^q - (\psi(t_2)-\psi(a))^q \leq 0$. Therefore:

$$\begin{aligned}
\|(\mathcal{T}(x))(t_2) - (\mathcal{T}(x))(t_1)\| &\leq 2\frac{\mu_f^* \phi(R)}{\Gamma(q+1)} (\psi(t_2)-\psi(t_1))^q \\
&= 2\mu_f^* \phi(R)(\psi(t_2)-\psi(t_1))^q.
\end{aligned}$$

As $t_2 \to t_1$, the right-hand side of the above inequality tends to zero independently of $x \in B_R$. Hence, we conclude that $\mathcal{T}(B_R) \subseteq C(\Delta,\ E)$ is bounded and equicontinuous.

Step 4. The Mönch's condition holds. For this purpose, let $\mathcal{V}$ be a subset of B_R, such that $\mathcal{V} \subset \overline{conv}(\mathcal{T}(\mathcal{V}) \cup \{0\})$, $\mathcal{V}$ is bounded and equicontinuous, and therefore, the function $\tau(t) = \mu((t))$ is continuous on Δ. By the properties of the Kuratowski measure of noncompactness, Lemma 2 and (H_3), we have:

$$\begin{aligned}
\tau(t) = \mu(\mathcal{V}(t)) &\leq \mu(\overline{conv}(\mathcal{T}(\mathcal{V})(t) \cup \{0\})) \leq \mu((\mathcal{V})(t)) \\
&\leq \mu \left\{ \int_a^t \frac{\psi'(s)(\psi(t)-\psi(s))^{q-1}}{\Gamma(q)} f(s,x(s)) \mathrm{d}s : x \in \mathcal{V} \right\} \\
&\leq \int_0^t \frac{\psi'(s)(\psi(t)-\psi(s))^{q-1}}{\Gamma(q)} \mu(f(s,\ \mathcal{V}_1(s))) \mathrm{d}s \\
&\leq \int_0^t \frac{\psi'(s)(\psi(t)-\psi(s))^{q-1}}{\Gamma(q)} \mu_f(s)\mu(\mathcal{V}(s)) \mathrm{d}s \\
&\leq L\mu_f^* \|\tau\|.
\end{aligned}$$

This gives that:

$$\|\tau\| \leq L\mu_f^*\|\tau\|.$$

By (7), it follows that $\|\tau\|_\infty = 0$; that is $\tau(t) = 0$ for each $t \in \Delta$. In the similar way, we have $\tau(t) = 0$. Hence, $\mu(\mathcal{V}(t)) \leq \mu(\mathcal{V}(t)) = 0$ and $\mu(\mathcal{V}(t)) \leq \mu(\mathcal{V}(t)) = 0$, this means that $\mathcal{V}(t)$ is relatively compact in X. In view of the AscoliArzelá theorem, $\mathcal{V}$ is relatively compact in B_R. By theorem 1, there is a fixed point x of $\mathcal{T}$ on B_R, which is a solution of (1.1). This completes the proof of theorem 2. □

5. Illustrative example

In this section we give an example to illustrate our main result.

Consider the following hybrid fractional differential equation:

$$\begin{cases} \frac{\partial^q}{\partial t^q} y(t,\ x) = \frac{\partial^2}{\partial x^2} y(t,x) \\ \qquad\qquad\qquad +f(t, y(t,x)),\ x \in [0,\pi],\ t \in (0,b],\ 0 < q < 1, \\ y(t,0) = y(t,\pi) = 0,\ t \in (0,T], \\ y(0,x) = y_0,\ x \in [0,\pi], \end{cases} \tag{1.11}$$

where $f:\ [0,T] \times \mathbb{R} \to \mathbb{R}$ is a given function. Let

$$u(t)(x) = y(t,x)\ ,\ t \in [0,T],\ x \in [0,\ \pi],$$
$$g(t,u)(x) = f(t,u(x))\ ,\ t \in [0,T],\ x \in [0,\pi].$$

We choose $X = C([\mathrm{O},\ \pi],\ \mathbb{R})$ endowed with the uniform topology and consider the operator $A : D(A) \subset X \to X$ defined by:

$$D(A) = \{u \in C^2([0,\ \pi],\ \mathbb{R}) : u(0) = u(\pi) = 0\},\ Au = u.$$

It is well known that the operator A satisfies the Hille-Yosida condition with $(0,\ +\infty) \subset \rho(A)$, $\|(\lambda I - A)^{-1}\| \leq \dfrac{1}{\lambda}$ for $\lambda > 0$, and

$$\overline{D(A)} = \{u \in X\ :\ u(0) = u(\pi) = 0\} \neq X.$$

6. Conclusion

In this paper, we have studied the existence of the solution of fractional differential equation of order $0 < q < 1$ of the form (1.1), the search for the solution and the results of the existence are based on: the transform of laplace, the measure of non compactness of Hausdorff and the theorem of Mönch. For the application, we proposed an example problem to test the theoretical results of this study.

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